Step-by-Step Green Function Method, Dirichlet

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There are many places in this method where it is easy to make mistakes if you are not very careful with the notation. For this reason, it is recommended that you use the notation and steps defined here. The sample problem shown below is for a real point charge Q centered over a rectangular plate at V in a grounded plane.

Steps	Sample Problem Showing Proper Notation
1. Write down mathematically the charge distribution of the *real problem* in terms of primed coordinates	$\rho(\mathbf{x}')=Q\delta(x')\delta(y')\delta(z'-z_0)$
2. Write down mathematically the boundary surface's shape and location on which the boundary condition exists	surface S is the plane at $z=0$
3. Write down mathematically the boundary condition of the *real problem* in terms of primed coordinates	$\Phi(\mathbf{x}')=\begin{Bmatrix} V \text{ if } \lvert x'\rvert<a \text{ and } \lvert y'\rvert<b \\ 0 \text{ if } \lvert x'\rvert>a \text{ or } \lvert y'\rvert>b \end{Bmatrix}$ on S
4. Create the *simpler problem's* charge distribution by placing a point charge at an <u>*arbitrary location*</u> defined in terms of primed coordinates...	q is at (x', y', z')
5. Create the *simpler problem's* boundary surface as the exact same as the *real problem's*	surface S is the plane at $z=0$
6. Create the *simpler problem's* boundary condition as a grounded conductor	$\Phi(\mathbf{x})=0$ on S
7. Solve the *simpler problem* by placing an image point charge(s) at a logical location(s) on the other side of the boundary surface from the original point charge, and removing the boundary surface	q' is at $(x', y', -z')$
8. Write down the *simple problem's* solution as the potential due to the point charge and its images charge(s)	$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\frac{q}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}+\frac{1}{4\pi\epsilon_0}\frac{q'}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$
9. Apply the *simpler problem's* boundary condition of a grounded conductor to determine the image charge(s) magnitude and location	$q'=-q$
10. Write out the final solution to the *simpler problem*, where the unprimed coordinates give the location of space where the potential is measured, and the primed coordinates give the location of the arbitrarily placed point charge	$\Phi(\mathbf{x})=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right]$
11. Convert the solution of the *simpler problem* to the Green function of the *real problem*: Φ becomes G q becomes $4\pi\varepsilon_0$ Make sure the Green function is symmetric between **x** and **x**'. If not, you have made a mistake that must be fixed.	$G(\mathbf{x},\mathbf{x}')=\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}-\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}$

12. For later use, find the partial derivative of the Green function, at the surface, in the direction normal to the surface and away from the volume of the *real problem* where we want to know the potential.	$\left[\frac{\partial G}{\partial n'}\right]_{\text{on } S}=\left[-\frac{\partial G}{\partial z'}\right]_{z'=0}$ $\left[\frac{\partial G}{\partial n'}\right]_{\text{on } S}=\frac{-2z}{((x-x')^2+(y-y')^2+z^2)^{3/2}}$
13. Write down the general form of the Green function solution to the *real problem*, making sure that integration is over primed variables, and the real charge distribution and real boundary condition are given in terms of primed variables.	$\Phi=\frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{x}')G\,d^3\mathbf{x}'-\frac{1}{4\pi}\oint\Phi(\mathbf{x}')\frac{\partial G}{\partial n'}da'$
14. Expand the integrals in the solution into the coordinates system that you are using	$\Phi=\frac{1}{4\pi\epsilon_0}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty\rho(\mathbf{x}')G\,dx'dy'dz'$ $-\frac{1}{4\pi}\int_{-\infty}^\infty\int_{-\infty}^\infty\Phi(\mathbf{x}')\frac{\partial G}{\partial n'}dx'dy'$
15. Plug the following into the integrals in the solution: - the real charge distribution ρ as defined in step 1 - the Green function G as found in step 11 - the real boundary condition Φ as defined in step 3 - the partial of the Green function as found in step 12	
16. Evaluate the Dirac deltas and integrals as much as possible.	$\Phi=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2+y^2+(z-z_0)^2}}-\frac{1}{\sqrt{x^2+y^2+(z+z_0)^2}}\right]$ $+\frac{Vz}{2\pi}\int_{-b}^{b}\int_{-a}^{a}\frac{1}{((x-x')^2+(y-y')^2+z^2)^{3/2}}dx'dy'$