Alternate Potentials, Electromagnetic Theory I

Dr. Christopher S. Baird
University of Massachusetts Lowell

1. Maxwell's Equations with General Potentials

- When we looked at putting Maxwell's equations in terms of potentials in the standard way, we found the magnetic field **B** was the curl of a vector field. But in magnetostatics, we discovered that we could define the **B** field in terms of the gradient of a scalar potential as well if there were no sources present. Can we use the same alternate approach in electrodynamics?
- The point is that the potentials are non-physical so we can define them however we want and we will still end up with the same answers for the fields. This is known as gauge freedom.
- The standard potential definitions may be the most useful mathematically, but they are not unique and they are not the most general.
- Let us write down Maxwell's equations with very general potential definitions and then see how they reduce to the standard form as a special case.
- Define:

$$\mathbf{E}=-\nabla\Phi_E-\frac{\partial\mathbf{A}_M}{\partial t}+\nabla\times\mathbf{A}_E \quad \text{and} \quad \mathbf{B}=\frac{1}{c^2}\nabla\Phi_M+\frac{1}{c^2}\frac{\partial\mathbf{A}_E}{\partial t}+\nabla\times\mathbf{A}_M$$

- Note that even though these definitions are much more general than the standard definitions, they are not uniquely general. There is no physical uniqueness to the potentials in classical electrodynamics (quantum theory has more to say, though). To get uniqueness mathematically, we must artificially impose additional constrains on the potentials (gauge conditions).
- Here Φ_E is the familiar electrostatic scalar potential in the electrostatic limit, $\mathbf{A}_M$ is the familiar magnetostatic vector potential in the static limit, Φ_M is the familiar magnetostatic scalar potential in the static limit in regions with no current, and $\mathbf{A}_E$ is an electrostatic vector potential added to make the equations symmetric.
- Plug these trial solutions into the Maxwell's equations to find (after several terms drop out because we always have $\nabla\cdot(\nabla\times\mathbf{A})=0$ and $\nabla\times(\nabla\Phi)=0$):

$$\left.\begin{aligned}
&\nabla^2\Phi_E=-\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}_M-\frac{\rho_{\text{total}}}{\epsilon_0}\\
&\nabla^2\Phi_M=-\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}_E\\
&\nabla(\nabla\cdot\mathbf{A}_E)-\nabla^2\mathbf{A}_E+\frac{1}{c^2}\frac{\partial^2\mathbf{A}_E}{\partial t^2}=-\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\Phi_M\\
&\nabla(\nabla\cdot\mathbf{A}_M)-\nabla^2\mathbf{A}_M+\frac{1}{c^2}\frac{\partial^2\mathbf{A}_M}{\partial t^2}=-\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\Phi_E+\mu_0\mathbf{J}_{\text{total}}
\end{aligned}\right\}\ \textit{Maxwell Equations with general potentials}$$

- Because we defined the potentials symmetrically, Maxwell's equations in potentials form end up partially symmetric. If magnetic charges and currents existed, the symmetry would be perfect.
- We now get a unique solution by imposing additional constraints on the potentials (choosing a gauge). For illustration purposes, let us choose the following gauges:

$$\nabla\cdot\mathbf{A}_M=-\frac{1}{c^2}\frac{\partial\Phi_E}{\partial t}\quad\text{and}\quad\mathbf{A}_E=0,\ \Phi_M=0\qquad\textit{Lorenz Gauge}$$

$$\nabla\cdot\mathbf{A}_E=-\frac{1}{c^2}\frac{\partial\Phi_M}{\partial t}\quad\text{and}\quad\mathbf{A}_M=0,\ \Phi_E=0\qquad\textit{Alternate Lorenz Gauge}$$

$$\nabla\cdot\mathbf{A}_M=0\quad\text{and}\quad\mathbf{A}_E=0,\ \Phi_M=0\qquad\textit{Coulomb Gauge}$$

$$\nabla\cdot\mathbf{A}_E=0\quad\text{and}\quad\mathbf{A}_M=0,\ \Phi_E=0\qquad\textit{Alternate Coulomb Gauge}$$

- Each set of constraints leads to a particular form of Maxwell's equations, as shown in the next page.
- The symmetry between the **E** and **B** field is preserved even in the potentials representations. They become perfectly symmetric in regions with no sources.
- Note that the Alternate Lorenz Gauge and the Alternate Coulomb Gauge is only possible in regions without electric charges or electric currents. For this reason, they are rarely used in practice.
- By symmetry, the traditional Lorenz gauge is only possible in regions where there is no magnetic charges or magnetic currents (which happens to be the entire known universe).
- Now we see that the static magnetic scalar potential we were using in certain cases in magnetostatics is a special case of the Alternate Coulomb Gauge.

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi_E = -\frac{\rho_{\text{total}}}{\epsilon_0}$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}_M = -\mu_0\,\mathbf{J}_{\text{total}}$$

Lorenz Gauge

$$\text{where } \mathbf{E} = -\nabla\Phi_E - \frac{\partial\mathbf{A}_M}{\partial t} \text{ and } \mathbf{B} = \nabla\times\mathbf{A}_M$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\Phi_M = 0$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}_E = 0$$

Alternate Lorenz Gauge

$$\rho_{\text{total}} = 0 \quad \text{and} \quad \mathbf{J}_{\text{total}} = 0$$

$$\text{where } \mathbf{E} = \nabla\times\mathbf{A}_E \text{ and } \mathbf{B} = \frac{1}{c^2}\nabla\Phi_M + \frac{1}{c^2}\frac{\partial\mathbf{A}_E}{\partial t}$$

$$\nabla^2\Phi_E = -\frac{\rho_{\text{total}}}{\epsilon_0}$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}_M = \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\Phi_E - \mu_0\,\mathbf{J}_{\text{total}}$$

Coulomb Gauge

$$\text{where } \mathbf{E} = -\nabla\Phi_E - \frac{\partial\mathbf{A}_M}{\partial t} \text{ and } \mathbf{B} = \nabla\times\mathbf{A}_M$$

$$\nabla^2\Phi_M = 0$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}_E = \frac{1}{c^2}\nabla\frac{\partial\Phi_M}{\partial t}$$

Alternate Coulomb Gauge

$$\rho_{\text{total}} = 0 \quad \text{and} \quad \mathbf{J}_{\text{total}} = 0$$

$$\text{where } \mathbf{E} = \nabla\times\mathbf{A}_E \text{ and } \mathbf{B} = \frac{1}{c^2}\nabla\Phi_M + \frac{1}{c^2}\frac{\partial\mathbf{A}_E}{\partial t}$$