Lecture 9 Notes, Electromagnetic Theory I

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1. Electrostatic Equations with Ponderable Materials

- We desire to find the electric fields of any system which includes materials.
- We can consider a material to be a collection of small regions with a charge distribution.
- Let us find the potential due to one of these charge distributions, then integrate over all charge distributions to get the total potential.
- We know that we can expand the potential due to a localized charge distribution into its mulitpole moment contributions and only keep the first few terms if we are far away. Because we will shrink the charge regions to a very small size when forming the integral, any point in space can be considered far away from the charge region.
- Consider a small volume dV containing a charge density producing a potential $d\Phi$ expanded into multipole contributions:

$$d\,\Phi=\frac{1}{4\pi\epsilon_0}\frac{dq}{r}+\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{x}}{r^3}+\ldots$$

- Because the charge region is infinitesimal and we want a macroscopic expression, we are far enough away that terms become negligible except the monopole and dipole terms.
- We switch to absolute coordinates to allow us to add up effects at different locations.

$$d\,\Phi=\frac{1}{4\pi\epsilon_0}\frac{dq}{|\mathbf{x}-\mathbf{x}'|}+\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}$$

- Multiply and divide by the volume dV:

$$d\,\Phi=\frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{x}-\mathbf{x}'|}\left(\frac{dq}{dV}\right)dV+\frac{1}{4\pi\epsilon_0}\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\cdot\left(\frac{\mathbf{p}}{d\,V}\right)dV$$

- By definition, the charge per unit volume is the charge density, $\frac{dq}{dV}=\rho(\mathbf{x}')$, and the average dipole moment per unit volume is the polarization, $\frac{\mathbf{p}}{dV}=\mathbf{P}$.

$$d\,\Phi=\frac{1}{4\pi\epsilon_0}\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}dV+\frac{1}{4\pi\epsilon_0}\frac{\mathbf{P}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}dV$$

- To get the total electric potential, we add up the effects of all the small charge regions. Mathematically, this means integrating both sides of the equation:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'+\frac{1}{4\pi\epsilon_0}\int\frac{\mathbf{P}\cdot(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}'$$

- This is now the potential due to all charge regions in the material. Because we started very

generally, it includes all materials (except materials where quadrupole moments, etc. are significant). The first term is just Coulomb's law with no materials telling us the potential due to the free/excess charges. The second term is the potential due to the material itself. Typically, an applied field induces the polarization **P** in the material, which then creates its own field.
- We want to combine the two terms.

- Use the mathematical identity: $\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}=\nabla' \frac{1}{|\mathbf{x}-\mathbf{x}'|}$

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'+\frac{1}{4\pi\epsilon_0}\int\mathbf{P}\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

- Apply integration by parts to the second integral:

$$\int\mathbf{P}\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'=\left[\frac{\mathbf{P}\cdot\mathbf{n}}{|\mathbf{x}-\mathbf{x}'|}\right]_S-\int\frac{\nabla'\cdot\mathbf{P}}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

- We are integrating over all space, so the bounding surface is at infinity where there is no material. This makes the first term vanish, so that:

$$\int\mathbf{P}\cdot\nabla'\frac{1}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'=-\int\frac{\nabla'\cdot\mathbf{P}}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

- Use this identity in the potential equation:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'-\frac{1}{4\pi\epsilon_0}\int\frac{\nabla'\cdot\mathbf{P}}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

- Collect terms:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{(\rho(\mathbf{x}')-\nabla'\cdot\mathbf{P})}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

- This is just Coulomb's law for the total potential if we recognize the term in parentheses as the total charge density:

$$\boxed{\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{\rho_{\text{total}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'}\quad\text{where}\quad\boxed{\rho_{\text{total}}(\mathbf{x}')=\rho(\mathbf{x}')+\rho_{\text{pol}}(\mathbf{x}')}\quad\text{and}\quad\boxed{\rho_{\text{pol}}(\mathbf{x}')=-\nabla'\cdot\mathbf{P}}$$

- If we divide the rightmost equation directly above by ε_0 it looks like Gauss's law in differential form:

$$\nabla'\cdot(-\mathbf{P}/\epsilon_0)=\frac{1}{\epsilon_0}\rho_{\text{pol}}(\mathbf{x}')$$

so that we can identify (**-P**/ε_0) as the electric field caused by the polarization charge density.
- The quantity ρ_{pol} is the “polarization charge density”, or the “bound charge density”, or the

"induced charge density". It is the material's response to the fields.
- The quantity ρ is the "free charge density", or more accurately the "excess charge density", or the "inducing charge density". It is the charge that is placed when the problem is created (e.g. charging up a balloon by rubbing it in your hair). This is the charge we have already dealt with.
- The quantity ρ_{total} is the total charge density, which is the sum of the bound and free charges.
- It should be noted that, because of the way it has been defined, the polarization (the average dipole moment density) **P** points in the opposite direction from the field due to the polarization:

Material response electric field = $(-\mathbf{P}/\varepsilon_0)$

- This is because the dipole moment of a charge region always points from its negative side to its positive side, whereas electric fields start on positive charges and end on negative charges.
- While possibly confusing, this definition is useful because dipoles are seen to line up with the applied field.
- All of our results from electrostatics without materials carry over now if we realize that we must use the total charge distribution ρ_{total} when we seek the total field **E**.
- Gauss's law in differential form becomes:

$$\nabla\cdot\mathbf{E}=\frac{1}{\epsilon_0}\rho_{\text{total}}(\mathbf{x})$$

$$\nabla\cdot\mathbf{E}=\frac{1}{\epsilon_0}(\rho(\mathbf{x})+\rho_{\text{pol}}(\mathbf{x}))$$

$$\nabla\cdot\mathbf{E}=\frac{1}{\epsilon_0}(\rho(\mathbf{x})-\nabla\cdot\mathbf{P})$$

- The field **E** is the total field, and the quantity $(-\mathbf{P}/\varepsilon_0)$ is the induced field, so that the last piece in the equation above is a field-like entity and can be loosely thought of as the original externally applied field plus interactions.
- Define this applied field as $(\mathbf{D}/\varepsilon_0)$ where **D** is called the displacement. Because $(\mathbf{D}/\varepsilon_0)$ is an electric-field-like entity it must obey its version of Gauss's Law:

$$\nabla\cdot\left(\frac{\mathbf{D}}{\epsilon_0}\right)=\frac{1}{\epsilon_0}\rho(\mathbf{x})$$

$$\boxed{\nabla\cdot\mathbf{D}=\rho(\mathbf{x})}$$

- Plugging this into the equation above:

$$\nabla\cdot\mathbf{E}=\frac{1}{\epsilon_0}(\nabla\cdot\mathbf{D}-\nabla\cdot\mathbf{P})$$

- Integrate both sides:

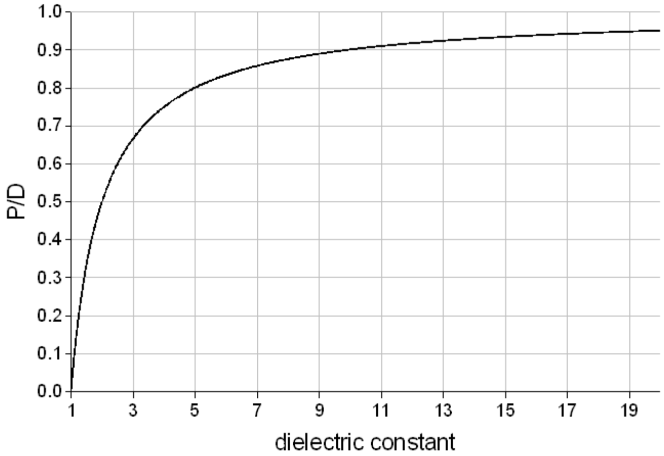
$$\boxed{\mathbf{E}=\mathbf{D}/\epsilon_0-\mathbf{P}/\epsilon_0}$$

- This simple equation states that the total electric field **E** is the sum of the applied field plus interactions ($\mathbf{D}/\varepsilon_0$) and the resulting induced field ($-\mathbf{P}/\varepsilon_0$). While this statement is true, we must be careful about what we mean. The free charges and bound charges interact with each other so that they can end up in different locations than where originally placed. For this reason, **D** is more than just what the field would be in the absence of the material.
- All of the equation thus far apply to any ponderable material where the dipole moments are the dominant terms in the response of the material to the applied field.
- These equations are not particularly useful until an explicit model is developed allowing us to find the polarization **P** based on the material's static properties.

2. Electrostatic Equations in Linear Isotropic Dielectric Materials

- The simplest and and most common type of dielectric material is that of linear, isotropic dielectrics.
- The isotropic property means that the applied field will always give rise to the same induced field, no matter how the material is oriented.
- The linear property means that the induced field is linearly proportional to the applied field inducing it. In other words, the polarization **P** points in the same direction as the displacement **D** and the magnitude of the polarization equals some constant times the magnitude of the displacement:

$$\boxed{\mathbf{P}=\frac{\epsilon_r-1}{\epsilon_r}\mathbf{D}}$$



- The constant linking the two is chosen to have this form to simplify the final equations, as is shown below.
- The variable ε_r is known as the *relative electric permittivity* of a material, or its *dielectric constant*. It is defined relative to the permittivity of free space: $\epsilon_r=\epsilon/\epsilon_0$.
- Most materials have dielectric constants between 1 and 100.
- Vacuum has a dielectric constant of 1 by definition, so that there is no polarization in response the applied electric field.
- Silicon Dioxide (sand or quartz) has a dielectric constant of 3.7, so that the polarization has about 73% the strength of the displacement.
- Water at room temperature has a static dielectric constant of 80, so that the polarization is 99% the magnitude of the displacement. In other words, most of the applied electric field is canceled out by the responding polarization field. That is why radio wave cannot penetrate water. (Higher frequencies of light do penetrate water, but that is because light is really an electrodynamic and not an electrostatic phenomenon, which becomes more apparent at higher frequencies.)
- Now combine the linear material response equation with the electrostatic field equations.

$\mathbf{E}=\mathbf{D}/\epsilon_0-\mathbf{P}/\epsilon_0$ and $\mathbf{P}=\frac{\epsilon_r-1}{\epsilon_r}\mathbf{D}$ for linear materials becomes:

$$\mathbf{E}=\left(1/\epsilon_0-1/\epsilon_0\frac{\epsilon_r-1}{\epsilon_r}\right)\mathbf{D}$$

$$\boxed{\mathbf{E}=(1/\epsilon)\mathbf{D}}$$

- Knowing this relationship for linear materials, we can simplify Gauss' Law for the displacement:

$$\nabla\cdot\mathbf{D}=\rho(\mathbf{x})$$

$$\boxed{\nabla\cdot(\epsilon\,\mathbf{E})=\rho(\mathbf{x})}$$

Gauss's Law in differential form inside a linear dielectric

- This equation applies everywhere, even if the material is nonuniform or we have many materials put together, as long as all of the materials are linear.
- If the material varies spatially, the permittivity may also vary spatially and can not be moved outside the divergence operator. The equation must be solved in this form.
- A special case is if the permittivity is uniform in a region and thus does not depend on position.
- Then the permittivity can be moved outside the divergence operator:

$$\boxed{\nabla\cdot\mathbf{E}=\frac{1}{\epsilon}\rho(\mathbf{x})}$$

Gauss's Law in differential form inside a linear uniform dielectric

- This equation states essentially that the divergence of the electric field **E** is equal to the constant effects of the material ($1/\varepsilon$) times the effect of the free charge density.
- Therefore, the greater the permittivity of the material (the more able the material is to be polarized), the less a charge density is able to create a total electric field.
- If the region containing the uniform dielectric material is taken to include all space, this equation applies everywhere. All the problems in such a medium reduce to the same problems done in previous chapters except the permittivity of free space ε_0 is replaced by the permittivity of the material ε.
- If the problem involves a few different regions with different materials, but each material is linear and uniform, then this equation applies separately in each region. In such a case the different regions will need to be solved independently and connected by boundary conditions.

3. Boundary-Value Problems with Dielectrics

- The last equation above only applies to regions of uniform dielectric.
- However, this equation can still be used for problems with regions with different materials if each region of uniform dielectric is solved independently and then the solutions are forced to match up at the boundaries.
- The first boundary condition is found by drawing a pillbox around the surface in the usual way, but only including the free charge and thus applying the integral to only the applied field:

$$\boxed{(\mathbf{D}_2-\mathbf{D}_1)\cdot\mathbf{n}_{12}=\sigma}$$

- Here $\mathbf{n}_{12}$ is the boundary surface normal pointing from region 1 into region 2.
- The second boundary condition is found by drawing a loop half-way in the surface:

$$\boxed{(\mathbf{E}_2 - \mathbf{E}_1) \times \mathbf{n}_{12} = 0}$$

- Consider two semi-infinite regions of linear uniform dielectric material that meet at the *x*-*y* plane.
- The material on the negative *z* side, in region 2, has a permittivity ε_2 and the material on the positive *z* side, in region 1, has a permittivity ε_1.
- A point charge *q* is embedded on the *z*-axis a distance *d* from the origin in region 1.
- The equations to be solved are:

$$\epsilon_1 \nabla \cdot \mathbf{E}_1 = \rho \quad \text{for } z > 0$$

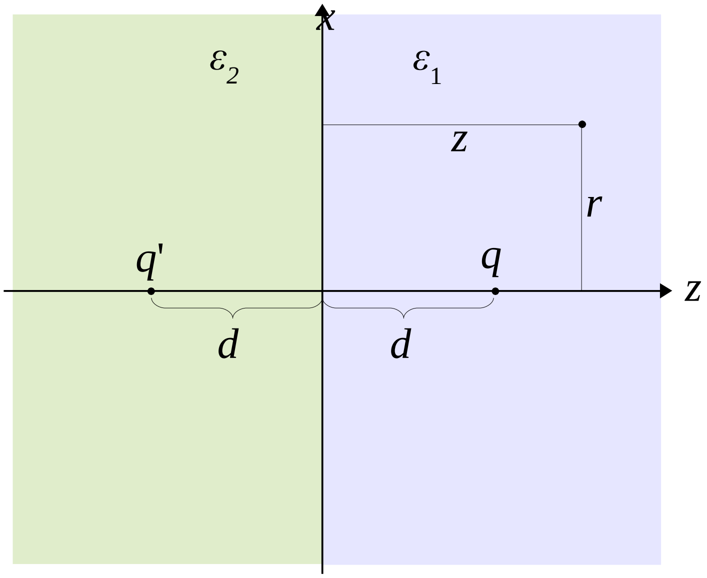
$$\epsilon_2 \nabla \cdot \mathbf{E}_2 = 0 \quad \text{for } z < 0$$

- The boundary conditions at $z = 0$ are:

$\epsilon_1 E_{1,z} = \epsilon_2 E_{2,z}$, and $E_{1,r} = E_{2,r}$ for all θ in cylindrical coordinates

- Let us use the image method to solve for the field in region 1.

- We place an image charge *q*' at $z = -d$ as shown in the image below:



- The potential in region 1 is just that felt by a point charge and its image. In cylindrical coordinates:

$$\Phi_1=\frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{r^2+(z-d)^2}}+\frac{q'}{\sqrt{r^2+(z+d)^2}}\right)$$

- To solve for the field in region 2, we treat it as a separate problem and place an image charge q'' at $z = d$. There are no other charges involved (the actual charge q does not directly come into play in region 2 because of the way we have set up the problem) so that the solution is:

$$\Phi_2=\frac{1}{4\pi\epsilon_2}\left(\frac{q''}{\sqrt{r^2+(z-d)^2}}\right)$$

- Now apply the boundary condition $\epsilon_1 E_{1,z}=\epsilon_2 E_{2,z}$ at $z = 0$:

$$\epsilon_1\frac{\partial}{\partial z}\Phi_1=\epsilon_2\frac{\partial}{\partial z}\Phi_2$$

$$\epsilon_1\frac{\partial}{\partial z}\left(\frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{r^2+(z-d)^2}}+\frac{q'}{\sqrt{r^2+(z+d)^2}}\right)\right)=\epsilon_2\frac{\partial}{\partial z}\left(\frac{1}{4\pi\epsilon_2}\left(\frac{q''}{\sqrt{r^2+(z-d)^2}}\right)\right)$$

$$\frac{\partial}{\partial z}\left(\frac{q}{\sqrt{r^2+(z-d)^2}}+\frac{q'}{\sqrt{r^2+(z+d)^2}}\right)=\frac{\partial}{\partial z}\left(\frac{q''}{\sqrt{r^2+(z-d)^2}}\right)$$

$$(-1/2)\frac{q(-2d)}{(r^2+d^2)^{3/2}}+(-1/2)\frac{q'(2d)}{(r^2+d^2)^{3/2}}=(-1/2)\frac{q''(-2d)}{(r^2+d^2)^{3/2}}$$

$$\boxed{q-q'=q''}$$

- Apply the boundary condition: $E_{1,r}=E_{2,r}$ at $z = 0$:

$$\frac{\partial}{\partial r}\Phi_1=\frac{\partial}{\partial r}\Phi_2$$

$$\frac{\partial}{\partial r}\left(\frac{1}{4\pi\epsilon_1}\left(\frac{q}{\sqrt{r^2+(z-d)^2}}+\frac{q'}{\sqrt{r^2+(z+d)^2}}\right)\right)=\frac{\partial}{\partial r}\left(\frac{1}{4\pi\epsilon_2}\left(\frac{q''}{\sqrt{r^2+(z-d)^2}}\right)\right)$$

$$\left(\frac{1}{4\pi\epsilon_1}\left((-1/2)\frac{2rq}{(r^2+d^2)^{3/2}}+(-1/2)\frac{2rq'}{(r^2+d^2)^{3/2}}\right)\right)=\left(\frac{1}{4\pi\epsilon_2}\left((-1/2)\frac{2rq''}{(r^2+d^2)^{3/2}}\right)\right)$$

$$\boxed{\frac{1}{\epsilon_1}(q+q')=\frac{1}{\epsilon_2}q''}$$

The two boxed equations can be solved in terms of the known charge q.

$$q'=\frac{\epsilon_1-\epsilon_2}{\epsilon_1+\epsilon_2}q$$

$$q'' = \frac{2\epsilon_2}{\epsilon_1+\epsilon_2} q$$

- So that the final solution for the potential is:

$$\boxed{\Phi = \frac{q}{4\pi}\left(\frac{1}{\epsilon_1\sqrt{r^2+(z-d)^2}} + \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1(\epsilon_1+\epsilon_2)}\right)\frac{1}{\sqrt{r^2+(z+d)^2}}\right)} \quad \text{if } z > 0$$

$$\boxed{\Phi = \frac{q}{4\pi}\left(\frac{2}{\epsilon_1+\epsilon_2}\right)\frac{1}{\sqrt{r^2+(z-d)^2}}} \quad \text{if } z < 0$$

- Drawing a pillbox in the usual way, but only enclosing the polarization charge, leads to the boundary condition:

$$\sigma_{\text{pol}} = -(\mathbf{P}_2 - \mathbf{P}_1)\cdot\mathbf{n}_{12} \quad \text{at } z = 0$$

- For linear materials:

$$\sigma_{\text{pol}} = -\left(\frac{\epsilon_{\text{r},2}-1}{\epsilon_{\text{r},2}}\mathbf{D}_2 - \frac{\epsilon_{\text{r},1}-1}{\epsilon_{\text{r},1}}\mathbf{D}_1\right)\cdot\mathbf{n}_{12}$$

$$\sigma_{\text{pol}} = -\left(\frac{\epsilon_{\text{r},2}-1}{\epsilon_{\text{r},2}}\epsilon_2\mathbf{E}_2 - \frac{\epsilon_{\text{r},1}-1}{\epsilon_{\text{r},1}}\epsilon_1\mathbf{E}_1\right)\cdot\mathbf{n}_{12}$$

$$\sigma_{\text{pol}} = -\left((\epsilon_2-\epsilon_0)\mathbf{E}_2 - (\epsilon_1-\epsilon_0)\mathbf{E}_1\right)\cdot\mathbf{n}_{12}$$

$$\sigma_{\text{pol}} = -\left((\epsilon_2-\epsilon_0)(-\nabla\Phi_2) - (\epsilon_1-\epsilon_0)(-\nabla\Phi_1)\right)\cdot\mathbf{n}_{12}$$

$$\sigma_{\text{pol}} = (\epsilon_2-\epsilon_0)\left(-\frac{\partial}{\partial z}\Phi_2\right) - (\epsilon_1-\epsilon_0)\left(-\frac{\partial}{\partial z}\Phi_1\right)$$

$$\sigma_{\text{pol}} = (-\epsilon_2+\epsilon_0)\frac{\partial}{\partial z}\frac{q}{4\pi}\left(\frac{2}{\epsilon_1+\epsilon_2}\right)\frac{1}{\sqrt{r^2+(z-d)^2}}$$
$$+(\epsilon_1-\epsilon_0)\frac{\partial}{\partial z}\frac{q}{4\pi}\left(\frac{1}{\epsilon_1\sqrt{r^2+(z-d)^2}} + \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1(\epsilon_1+\epsilon_2)}\right)\frac{1}{\sqrt{r^2+(z+d)^2}}\right)$$

$$\sigma_{\text{pol}} = (-\epsilon_2+\epsilon_0)\frac{q}{4\pi}\left(\frac{2}{\epsilon_1+\epsilon_2}\right)(-1/2)\frac{(-2d)}{(r^2+d^2)^{3/2}}$$
$$+(\epsilon_1-\epsilon_0)\frac{q}{4\pi}\left((-1/2)\frac{(-2d)}{\epsilon_1(r^2+d^2)^{3/2}} + \left(\frac{\epsilon_1-\epsilon_2}{\epsilon_1(\epsilon_1+\epsilon_2)}\right)(-1/2)\frac{(2d)}{(r^2+d^2)^{3/2}}\right)$$

$$\boxed{\sigma_{\text{pol}} = -\frac{q}{2\pi}\frac{\epsilon_0(\epsilon_2-\epsilon_1)}{\epsilon_1(\epsilon_2+\epsilon_1)}\frac{d}{(r^2+d^2)^{3/2}}}$$

4. Electrostatic Energy in Dielectric Media

- As derived in a previous lecture, the electrostatic potential energy stored in a charge configuration in vacuum is given by:

$$W = \frac{1}{2}\int \rho(\mathbf{x})\Phi(\mathbf{x})\,d\,\mathbf{x}$$

- This equation can still be used in the presence of *linear* dielectrics if we think of assembling the free charge density ρ, where each piece interacts with the *total* potential Φ generated by the charge density, which includes the dielectric effects.
- The free charge density ρ gives rise just to the applied electric field (proportional to the displacement D) according to:

$$\nabla\cdot\mathbf{D} = \rho(\mathbf{x})$$

so that we have:

$$W = \frac{1}{2}\int \nabla\cdot\mathbf{D}\,\Phi\,d\,\mathbf{x}$$

Integration by parts leads to:

$$W = -\frac{1}{2}\int \mathbf{D}\cdot\nabla\Phi\,d\,\mathbf{x}$$

The total potential is related to the total electric field according to $\mathbf{E} = -\nabla\Phi$. The energy is:

$$\boxed{W = \frac{1}{2}\int \mathbf{E}\cdot\mathbf{D}\,d\,\mathbf{x}}$$

For linear materials, $\mathbf{E} = (1/\epsilon)\mathbf{D}$ so that we can cast the energy in the form:

$$\boxed{W = \frac{\epsilon}{2}\int |\mathbf{E}|^2\,d\,\mathbf{x}}$$