Lecture 8 Notes, Electromagnetic Theory I

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1. Multipole Expansion of the Potential

- Consider a localized charge density completely contained within some region *R*.
- *Very* far away from the region *R*, the charge density behaves more and more like a sphere or a point charge.
- *Far* away from the region *R* then we can make an expansion of the potential in spherical harmonics and keep only the first few terms and it will still be a valid approximation to the solution.
- This is useful when the charge density is localized but too complex to be approached in an exact way.
- Because we want the potential far away from the charge density, where there is no charge, we can use the spherical coordinates solution to the Laplace equation when a valid solution is required on the full azimuthal range:

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_{lm}r^{l}+B_{lm}r^{-l-1}\right)Y_{lm}(\theta,\phi)$$

where $Y_{lm}(\theta,\phi)=\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta)$ are the spherical harmonics

- The region we are interested in includes infinity, but not the origin. To ensure the solution approaches zero at infinity, we require $A_l = 0$. The solution now becomes:

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}B_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

- For later convenience, we redefine the arbitrary constant, $B_{lm}=\frac{1}{4\pi\epsilon_0}\frac{4\pi}{2l+1}q_{lm}$ so that:

$$\boxed{\Phi(r,\theta,\phi)=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}}$$

- This equation is called a multipole expansion. The $l = 0$ term is called the monopole term, $l = 1$ are the dipole terms, etc.
- We must now determine the coefficients q_{lm} to fully solve the problem.
- The solution in integral form was already obtained as Coulomb's law for the potential:

$$\Phi=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

- We expand the $1/|\mathbf{x}-\mathbf{x}'|$ factor into spherical harmonics, remembering that we are interested in the solution far away from the charge so that we want the $x > x'$ case.

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{r'^{l}}{r^{l+1}}\frac{1}{2l+1}Y^{*}_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)$$

so that:

$$\Phi=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\left[\int Y^{*}_{lm}(\theta',\phi')r'^{l}\rho(\mathbf{x}')d\mathbf{x}'\right]\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

- Comparing this solution to the one above, it becomes apparent that:

$$\boxed{q_{lm}=\int Y^{*}_{lm}(\theta',\phi')r'^{l}\rho(\mathbf{x}')d\mathbf{x}'}$$

- These coefficients are called the spherical multipole moments. Their physical significance can be seen by representing the first few terms explicitly in Cartesian coordinates.
- The $l = 0$ term is just proportional to the total charge q, which is known as the monopole moment, and has no angular dependence:

$$q_{00}=\sqrt{\frac{1}{4\pi}}\int\rho(\mathbf{x}')d\mathbf{x}'$$

$$\boxed{q_{00}=\sqrt{\frac{1}{4\pi}}q}$$

- The $l = 1$ terms are proportional to the components of the electric dipole moment **p**.

$$q_{10}=\sqrt{\frac{3}{4\pi}}\int\cos(\theta')r'\rho(\mathbf{x}')d\mathbf{x}'$$

$$q_{10}=\sqrt{\frac{3}{4\pi}}\int z'\rho(\mathbf{x}')d\mathbf{x}'$$

$$\boxed{q_{10}=\sqrt{\frac{3}{4\pi}}p_z}$$

$$q_{1,\pm1}=\mp\sqrt{\frac{3}{8\pi}}\int\sin(\theta')e^{\mp i\phi'}r'\rho(\mathbf{x}')d\mathbf{x}'$$

$$q_{1,\pm1}=\mp\sqrt{\frac{3}{8\pi}}\int\sin(\theta')(\cos(\phi')\mp i\sin(\phi'))r'\rho(\mathbf{x}')d\mathbf{x}'$$

$$q_{1,\pm1}=\mp\sqrt{\frac{3}{8\pi}}\int(x'\mp iy')\rho(\mathbf{x}')d\mathbf{x}'$$

$$\boxed{q_{1,\pm 1} = \mp\sqrt{\frac{3}{8\pi}}(p_x \mp i\, p_y)}$$

- The total Cartesian dipole moment is defined as $\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')d\,\mathbf{x}'$
- The l = 2 terms are proportional to the Cartesian quadrupole moments Q_{ij}:

$$q_{20} = \sqrt{\frac{5}{16\pi}}\int (3\cos^2(\theta') - 1)r'^2\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$q_{20} = \sqrt{\frac{5}{16\pi}}\int (3z'^2 - r'^2)\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$\boxed{q_{20} = \sqrt{\frac{5}{16\pi}}Q_{33}}$$

$$q_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\int \sin(\theta')\cos(\theta')e^{\mp i\phi'}r'^2\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$q_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\int \sin(\theta')\cos(\theta')(\cos(\phi') \mp i\sin(\phi'))r'^2\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$q_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\int z'(x' \mp i\,y')\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$q_{2,\pm 1} = \mp\sqrt{\frac{15}{8\pi}}\int z'(x' \mp i\,y')\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$\boxed{q_{2,\pm 1} = \mp\frac{1}{3}\sqrt{\frac{15}{8\pi}}(Q_{13} \mp i\,Q_{23})}$$

$$q_{2,\pm 2} = \pm\sqrt{\frac{15}{32\pi}}\int \sin^2(\theta')e^{\mp i2\phi'}r'^2\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$q_{2,\pm 2} = \pm\sqrt{\frac{15}{32\pi}}\int \sin^2(\theta')(\cos(\phi') \mp i\sin(\phi'))^2 r'^2\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$q_{2,\pm 2} = \pm\sqrt{\frac{15}{32\pi}}\int (x' \mp i\,y')^2\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$\boxed{q_{2,\pm 2} = \frac{1}{3}\sqrt{\frac{15}{32\pi}}(Q_{11} \mp 2i\,Q_{12} - Q_{22})}$$

- In general, the Cartesian quadrupole moments are defined as:

$$Q_{ij}=\int(3x_i'x_j'-r'^2\delta_{ij})\rho(\mathbf{x}')d\mathbf{x}'$$

- With the first few coefficients found, we can write out the potential explicitly:

$$\Phi(r,\theta,\phi)=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

$$\begin{aligned}\Phi(r,\theta,\phi)=&\frac{1}{\epsilon_0 r}q_{0,0}Y_{0,0}\\&+\frac{1}{3\epsilon_0 r^2}\left[q_{1,0}Y_{1,0}+q_{1,1}Y_{1,1}+q_{1,-1}Y_{1,-1}\right]\\&+\frac{1}{5\epsilon_0 r^3}\left[q_{2,0}Y_{2,0}+q_{2,1}Y_{2,1}+q_{2,-1}Y_{2,-1}+q_{2,2}Y_{2,2}+q_{2,-2}Y_{2,-2}+...\right]\end{aligned}$$

- The spherical harmonics for these first few terms are simple enough to be written out explicitly:

$$\begin{aligned}\Phi(\mathbf{x})=&\frac{1}{\epsilon_0 r}\sqrt{\frac{1}{4\pi}}q_{0,0}\\&+\frac{1}{3\epsilon_0 r^2}\sqrt{\frac{3}{8\pi}}\left[\sqrt{2}q_{1,0}\cos(\theta)-(q_{1,1}e^{i\phi}-q_{1,-1}e^{-i\phi})\sin(\theta)\right]\\&+\frac{1}{5\epsilon_0 r^3}\sqrt{\frac{5}{16\pi}}\left[q_{2,0}(3\cos^2\theta-1)-\sqrt{6}(q_{2,1}e^{i\phi}-q_{2,-1}e^{-i\phi})\sin\theta\cos\theta+\sqrt{3/2}(q_{2,2}e^{i2\phi}+q_{2,-2}e^{-i2\phi})\sin^2(\theta)\right]+...\end{aligned}$$

- Now switch from spherical multipole moments to Cartesian multipole moments:

$$\begin{aligned}\Phi(\mathbf{x})=&\frac{q}{4\pi\epsilon_0 r}+\frac{1}{8\pi\epsilon_0 r^2}\left[2p_z\cos(\theta)+((p_x-ip_y)e^{i\phi}+(p_x+ip_y)e^{-i\phi})\sin(\theta)\right]\\&+\frac{1}{16\pi\epsilon_0 r^3}\left[Q_{33}(3\cos^2\theta-1)+2((Q_{13}-iQ_{23})e^{i\phi}+(Q_{13}+iQ_{23})e^{-i\phi})\sin\theta\cos\theta\right]\\&+\frac{1}{16\pi\epsilon_0 r^3}\left[1/2((Q_{11}-2iQ_{12}-Q_{22})e^{i2\phi}+(Q_{11}+2iQ_{12}-Q_{22})e^{-i2\phi})\sin^2(\theta)\right]+...\end{aligned}$$

$$\boxed{\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\left[\frac{q}{r}+\frac{\mathbf{p}\cdot\mathbf{x}}{r^3}+\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}Q_{ij}\frac{x_i x_j}{r^5}+...\right]}$$

- The last step was obtained using the traceless nature of the quadrupoles, i.e. $Q_{33}=-Q_{11}-Q_{22}$

2. Multipole Expansion of the Electric Field

- The electric field is most easily expressed in spherical coordinates.
- The potential in spherical coordinates was found to be:

$$\Phi(r,\theta,\phi)=\frac{1}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}q_{lm}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}$$

- The electric field is just the negative gradient:

$$\mathbf{E} = -\nabla \Phi$$

- In spherical coordinates:

$$\mathbf{E} = -\left[\hat{\mathbf{r}}\frac{\partial \Phi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial \Phi}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}\right]$$

$$\mathbf{E} = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{q_{lm}}{2l+1}\left[-\hat{\mathbf{r}}\frac{\partial}{\partial r}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} - \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial \theta}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}} - \hat{\boldsymbol{\phi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\frac{Y_{lm}(\theta,\phi)}{r^{l+1}}\right]$$

$$\boxed{\mathbf{E} = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{q_{lm}}{2l+1}\frac{1}{r^{l+2}}\left[\hat{\mathbf{r}}(l+1)Y_{lm}(\theta,\phi) - \hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta}Y_{lm}(\theta,\phi) - \hat{\boldsymbol{\phi}}\frac{i\,m}{\sin\theta}Y_{lm}(\theta,\phi)\right]}$$

- The monopole contribution to the electric field (l = 0) is then:

$$\mathbf{E}_{l=0} = \frac{1}{\epsilon_0}q_{00}\frac{1}{r^2}\left[Y_{00}(\theta,\phi)\right]\hat{\mathbf{r}}$$

$$\mathbf{E}_{l=0} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$

- This is, of course, the electric field due to a point charge q. This means that far enough away from a finite localized charge distribution with total charge q, the electric field is approximately equal to the field produced from a point charge q at its center.
- The dipole contributions to the electric field (l = 1) evaluate to:

$$\begin{aligned}\mathbf{E}_{l=1} = &\frac{1}{\epsilon_0}\frac{q_{1,-1}}{3}\frac{1}{r^3}\left[\hat{\mathbf{r}}\,2Y_{1,-1}(\theta,\phi) - \hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta}Y_{1,-1}(\theta,\phi) + \hat{\boldsymbol{\phi}}\frac{i}{\sin\theta}Y_{1,-1}(\theta,\phi)\right]\\ &+\frac{1}{\epsilon_0}\frac{q_{1,0}}{3}\frac{1}{r^3}\left[\hat{\mathbf{r}}\,2Y_{1,0}(\theta,\phi) - \hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta}Y_{1,0}(\theta,\phi)\right]\\ &+\frac{1}{\epsilon_0}\frac{q_{1,1}}{3}\frac{1}{r^3}\left[\hat{\mathbf{r}}\,2Y_{1,1}(\theta,\phi) - \hat{\boldsymbol{\theta}}\frac{\partial}{\partial \theta}Y_{1,1}(\theta,\phi) - \hat{\boldsymbol{\phi}}\frac{i}{\sin\theta}Y_{1,1}(\theta,\phi)\right]\end{aligned}$$

$$\begin{aligned}\mathbf{E}_{l=1} = &\left(\frac{1}{4\pi\epsilon_0 r^3}\right)\hat{\mathbf{r}}\left[2p_z\cos\theta + 2p_x\sin\theta\cos\phi + 2p_y\sin\theta\sin\phi\right]\\ &+\left(\frac{1}{4\pi\epsilon_0 r^3}\right)\hat{\boldsymbol{\theta}}\left[p_z\sin\theta - p_x\cos\phi\cos\theta - p_y\sin\phi\cos\theta\right]\\ &+\left(\frac{1}{4\pi\epsilon_0 r^3}\right)\hat{\boldsymbol{\phi}}\left[p_x\sin\phi - p_y\cos\phi\right]\end{aligned}$$

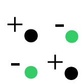
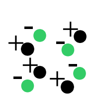
- After transforming every part of this equation into Cartesian coordinates and collecting terms, the dipole contribution of a localized charge distribution simplifies to a coordinate-independent form:

$$\boxed{\mathbf{E}_{l=1} = \frac{3\hat{\mathbf{x}}(\mathbf{p}\cdot\hat{\mathbf{x}}) - \mathbf{p}}{4\pi\epsilon_0|\mathbf{x}|^3}} \quad \text{where} \quad \boxed{\mathbf{p} = \int \mathbf{x}'\rho(\mathbf{x}')\,d\mathbf{x}'} \quad \text{(dipole at the origin)}$$

- The development thus far has assumed the multipoles are centered on the origin. If we desire to add together the effects of multiple charge distributions or multiple dipoles that are not at the same location, they can not both be at the origin. To generalize, the dipole at location $\mathbf{x}_0$ creates the field:

$$\boxed{\mathbf{E}_{l=1} = \frac{3(\widehat{\mathbf{x}-\mathbf{x}_0})(\mathbf{p}\cdot(\widehat{\mathbf{x}-\mathbf{x}_0})) - \mathbf{p}}{4\pi\epsilon_0|\mathbf{x}-\mathbf{x}_0|^3}} \quad \text{(dipole at } \mathbf{x}_0\text{)}$$

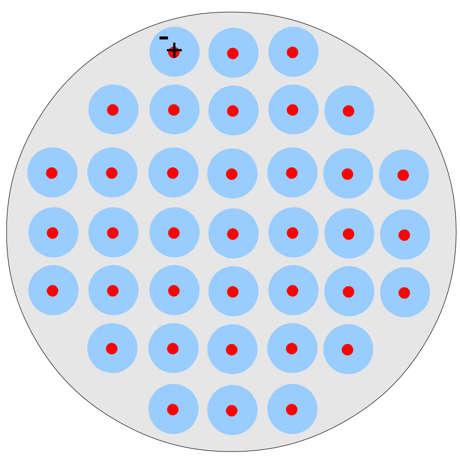
where the hat over the terms in parentheses means that the vector that results after taking the difference is normalized to be a unit vector.

Order	Name	Sample Point Form	Potential	Electric field	Spherical Moments	Cartesian Moments
$l = 0$	monopole		$\Phi \propto \frac{1}{r}$	$E \propto \frac{1}{r^2}$	q_{00}	q
$l = 1$	dipole		$\Phi \propto \frac{1}{r^2}$	$E \propto \frac{1}{r^3}$	$q_{1\text{-}1},\ q_{10},\ q_{11}$	$p_x,\ p_y,\ p_z$
$l = 2$	quadrupole		$\Phi \propto \frac{1}{r^3}$	$E \propto \frac{1}{r^4}$	$q_{2\text{-}2},\ q_{2\text{-}1},\ q_{20},\ q_{21},\ q_{22}$	$Q_{xx},\ Q_{xy},\ Q_{xz}$ $Q_{yx},\ Q_{yy},\ Q_{yz}$ $Q_{zx},\ Q_{zy},\ Q_{zz}$
$l = 3$	octupole		$\Phi \propto \frac{1}{r^4}$	$E \propto \frac{1}{r^5}$	$q_{3\text{-}3},\ q_{3\text{-}2},\ q_{3\text{-}1},$ $q_{30},\ q_{31},\ q_{32},\ q_{33}$	Q_{ijk} i, j, and $k = x, y$, or z
...						
l	l-pole	-	$\Phi \propto \frac{1}{r^{l+1}}$	$E \propto \frac{1}{r^{l+2}}$	$q_{l,-l},\ q_{l,-l+1},\ \ldots,\ q_{l,l-1},\ q_{l,l}$	$Q_{ijk\ldots l}$ $i, j, k\ldots l = x, y$, or z

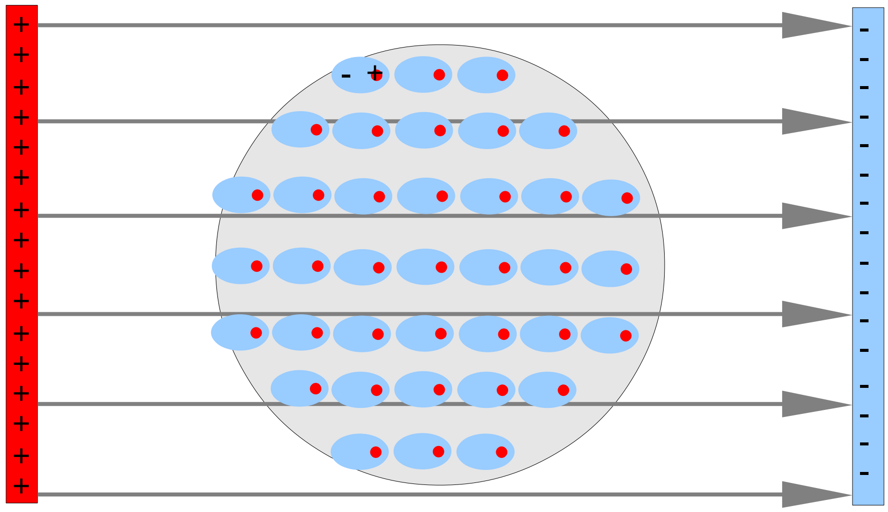
- In general, multipole moments depend on the choice of origin. A point charge at the origin has only a monopole moment, but move it off the axis and it has higher-order multiple moments. This is because we are measuring the potential as an expansion in spherical coordinates, which are defined from the origin.
- More specifically, the lowest nonvanishing multipole moment is always independent of the choice of origin, but the higher mutipoles do depend on the origin.

3. Basic Concepts about Electrostatics with Ponderable Media

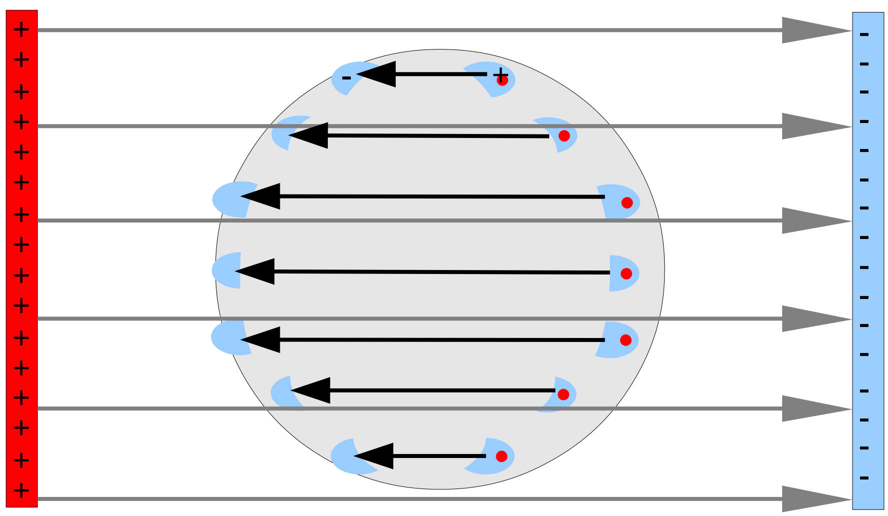
- Up to this point, all charges and fields have been assumed to be in vacuum, or in perfect conductors.
- Most materials are not perfect conductors, but are ponderable media which have some electrical response to the charges and fields. These effects must be taken into account.
- Macroscopically speaking, a non-conducting material contains a set of fixed positive and negative charge regions that typically cancel each other out on average, so that the material is electrically neutral.
- If extra charges are added to the material, it gains a net charge, but the majority of underlying charges still cancel out.
- When an eternal electric field is applied to the material, each charge region experiences a force from the field.
- Because the charge regions are fixed and cannot move in response to the field, they instead deform and gain non-zero multipole moments. The most dominant is the dipole.
- An electric field thus induces dipole moments in all of the charge regions, the total effect being that the electric field induces an electric polarization **P**(**x**) in the material.
- Consider the conceptual problem of a sphere of uniform material in a uniform electric field.
- Before the electric field is applied, the charge regions, and thus the material is neutral on average:



- After the electric field is applied, the negative portions of each charge region are attracted to the source of the electric field and deform. Each charge region has become polarized (gained a non-zero dipole moment, more or less aligned with the electric field).



- Deep inside the sphere, the charge regions still cancel each other out on average and the material has a net zero charge. On the edges of the sphere, however, there are no charges available to cancel out the deformed parts.



- There is then a net positive charge on one side of the sphere and a net negative charge on the other side of the sphere. (Note that this is a simplified conceptual picture. In reality, the originally uniform external field lines will be attracted to the surface charge and bend towards the sphere.)
- These polarization charges give rise to an electric field proportional to the polarization that opposes the original applied field.
- The total field (the actual field felt by some test charge placed in the material) will be some combination of the applied field and the induced field.
- In essence, the dielectric material weakens the effect of the applied field.

- For clarity, let us define the following:

D/ε_0: The *applied* electric field in the *absence* of the dielectric material (where **D** is called the displacement) plus interactions that are not directly attributable to the polarization
-P/ε_0: The *induced* electric field caused by the polarization of the material (where **P** is the polarization)
E: The *total* electric field including the applied field and the material's response

- The polarization **P** is defined as the macroscopically averaged dipole moment density. The induced field is the negative of the polarization. (This little curiosity arises from the fact that electric field lines are defined to point from positive to negative charges, whereas dipole vectors point from negative to positive charges.)
- Instead of treating the applied field as the result of charges external to the problem, we can explicitly include them. Let us then define the following:

ρ: The *free or excess charge* distribution, which gives rise to the applied electric field (**D**/ε_0)
ρ_{pol}: The *polarization charge*, (induced bound charge), which gives rise to the polarization **P**
ρ_{total}: The *total charge*, which is the sum of the free and polarization charge, which gives rise to the total field **E**.