Lecture 7 Notes, Electromagnetic Theory I

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1. Expansion of Green Functions in Spherical Coordinates

- Consider the problem of a spherical boundary with radius *a*, the potential is known on the boundary, there is charge present, and we wish to find the the potential anywhere *external* to the sphere.
- We have already solved this problem using Green's functions:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{x}')G_D d^3\mathbf{x}' - \frac{1}{4\pi}\oint\left(\Phi\frac{dG_D}{dn'}\right)da'$$

- External and internal to a spherical boundary we have already found the Green's function, it being the potential created by a unit charge and its image.

$$G(\mathbf{x},\mathbf{x}') = \frac{1}{|\mathbf{x}-\mathbf{x}'|} - \frac{1}{\left|\frac{x'}{a}\mathbf{x} - \frac{a}{x'}\mathbf{x}'\right|}$$

- Due to symmetry, this is valid for both the internal problem and the external problem. Let us call the first term G_1 and the second term G_2 and write them in terms of unit vectors.

$$G(\mathbf{x},\mathbf{x}') = G_1 + G_2 \quad \text{where} \quad G_1 = \frac{1}{|x\hat{\mathbf{x}} - x'\hat{\mathbf{x}}'|} \quad \text{and} \quad G_2 = -\frac{1}{\left|\left(\frac{x'x}{a}\right)\hat{\mathbf{x}} - (a)\hat{\mathbf{x}}'\right|}$$

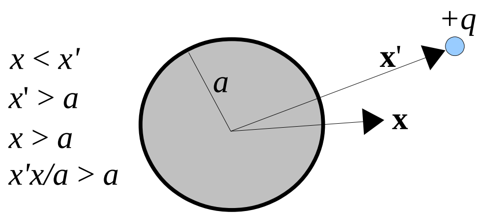
- When solving problems with both charges and boundary surfaces, the mathematics is simplified if the Green's function is expanded in spherical harmonics.
- The addition theorem was used to find the spherical harmonics expansion of a unit charge potential and is as follows:

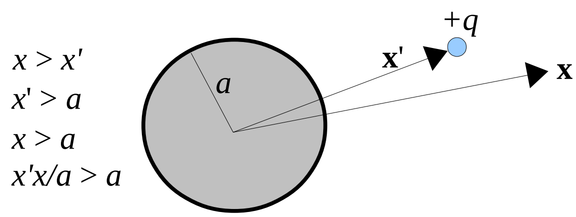
Expansion A: $$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{r_0^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \quad \text{if } r<r_0 \text{ and}$$

Expansion B: $$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r_0^l}{r^{l+1}}Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi) \quad \text{if } r>r_0$$

- Both terms in the Green function can be expanded into a series of spherical harmonics using these expansions, but we have to be careful about the different cases.

- *External* to the sphere, the source point x' is always greater than the sphere radius a, and the observation point x is always greater than a. This means that $x'x/a > a$ always, so that when expanding G_2, we always use the expansion B. But x is sometimes greater than x' and sometimes not, so when expanding G_1, we must handle the cases separately and use both expansions.



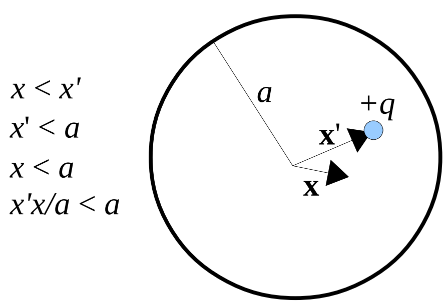


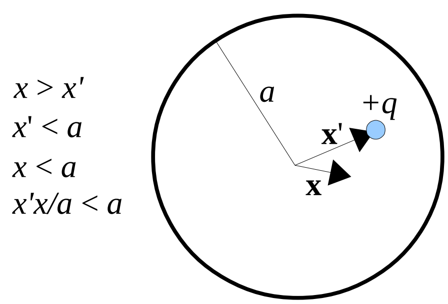
- Note that **x'** is the location of the real charge in the original problem, not the image charge. There is no image charge anymore. The image charge was just an intermediate mathematical trick to get the Green function. Using the right expansions for G_1 and G_2, we end up with:

$$G(\mathbf{x},\mathbf{x}')=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r^l}{r'^{l+1}}-\frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\quad \text{if } r<r' \text{ and}$$

$$G(\mathbf{x},\mathbf{x}')=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r'^l}{r^{l+1}}-\frac{1}{a}\left(\frac{a^2}{rr'}\right)^{l+1}\right]Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\quad \text{if } r>r'$$

- *Internal* to the sphere, now x and x' are always *less* than a. This amounts to using the expansion B for G_2 in both cases. When expanding G_1, we still have to take the cases separately and use both expansions.





- Using the right expansions for G_1 and G_2, we end up with the internal Green functions:

$$G(\mathbf{x},\mathbf{x}')=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r^l}{r'^{l+1}}-\frac{1}{a}\left(\frac{rr'}{a^2}\right)^{l}\right]Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\quad \text{if } r<r' \text{ and}$$

$$G(\mathbf{x},\mathbf{x}')=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r'^l}{r^{l+1}}-\frac{1}{a}\left(\frac{rr'}{a^2}\right)^{l}\right]Y^*_{lm}(\theta',\phi')Y_{lm}(\theta,\phi)\quad \text{if } r>r'$$

2. Solution of Potential Problems with the Spherical Green Function Expansions

- Consider a hollow grounded sphere of radius *b* containing a concentric ring of charge in the *x*-*y* plane with radius *a* and uniformly charged with a total charge *Q*. We wish to find the potential everywhere *inside* the sphere.
- This problem involves both a charge density:

$$\rho(\mathbf{x}')=\frac{Q}{2\pi a^2}\delta(r'-a)\delta(\cos\theta')$$

and a boundary condition $\Phi(r=b,\theta,\phi)=0$. We use the Green function solution to account for both.
- The Green's function solution for Dirichlet boundary conditions is:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{x}')G_D d^3\mathbf{x}'-\frac{1}{4\pi}\oint\left(\Phi\frac{dG_D}{dn'}\right)da'$$

- In this problem, the boundary condition $\Phi(r=b,\theta,\phi)=0$ eliminates the second term. The solution is now:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{x}')G_D d^3\mathbf{x}'$$

- The Green's function expansion for the interior-of-a-sphere problem was found above and is used to write out the potential explicitly:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\rho(\mathbf{x}')4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(rr')^l}{b^{2l+1}}\right]Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)d^3\mathbf{x}'$$

Plug in the charge density:

$$\Phi(\mathbf{x})=\frac{Q}{2\pi a^2\epsilon_0}\int_0^b\int_0^{2\pi}\int_0^{\pi}\delta(r'-a)\delta(\cos\theta')\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(rr')^l}{b^{2l+1}}\right]Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)r'^2\sin\theta' d\theta' d\phi' dr'$$

- The delta functions collapse two of the integrals:

$$\Phi(\mathbf{x})=\frac{Q}{2\pi a^2\epsilon_0}\int_0^{2\pi}\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(r_<r_>)^l}{b^{2l+1}}\right]Y_{lm}^*(\pi/2,\phi')Y_{lm}(\theta,\phi)a^2 d\phi'$$

where now $r_<$ and $r_>$ are the smaller and the larger of *r* and *a*.

- It is obvious from the symmetry that only the *m* = 0 term will contribute:

$$\Phi(\mathbf{x})=\frac{Q}{2\pi a^2\epsilon_0}\int_0^{2\pi}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(r_<r_>)^l}{b^{2l+1}}\right]Y_{l0}^*(\pi/2,\phi')Y_{l0}(\theta,\phi)a^2 d\phi'$$

$$\Phi(\mathbf{x})=\frac{Q}{2\pi a^2 \epsilon_0}\int_0^{2\pi}\sum_{l=0}^{\infty}\frac{1}{2l+1}\left[\frac{r_<^l}{r_>^{l+1}}-\frac{(r_< r_>)^l}{b^{2l+1}}\right]\sqrt{\frac{2l+1}{4\pi}}P_l(0)\sqrt{\frac{2l+1}{4\pi}}P_l(\cos\theta)a^2\,d\,\phi'$$

$$\Phi(\mathbf{x})=\frac{Q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}r_<^l\left[\frac{1}{r_>^{l+1}}-\frac{r_>^l}{b^{2l+1}}\right]P_l(0)\,P_l(\cos\theta)$$

- Use the fact that $P_{2n+1}(0)=0$ and $P_{2n}(0)=\frac{(-1)^n(2n-1)!!}{2^n n!}$:

$$\boxed{\Phi(\mathbf{x})=\frac{Q}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{2^n n!}r_<^{2n}\left[\frac{1}{r_>^{2n+1}}-\frac{r_>^{2n}}{b^{4n+1}}\right]P_l(\cos\theta)}$$