Lecture 5 Notes, Electromagnetic Theory I

Dr. Christopher S. Baird
University of Massachusetts Lowell

1. The Laplace Equation in Spherical Coordinates

- In this coordinate system, r is the radial distance from the origin to the observation point, θ is the polar angle that the point makes with the z-axis, and ϕ is the azimuthal angle in the x-y plane relative to the x-axis.
- Spherical coordinates are useful when the boundary conditions have a spherical shape or symmetry.
- The Laplace equation in spherical coordinates:

$$\nabla^2 \Phi = 0$$

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}=0$$

- Use the method of separation of variables by trying a solution of the form:

$$\Phi(r,\theta,\phi)=\frac{R(r)}{r}P(\theta)Q(\phi)$$

Here an extra factor ($1/r$) is included to anticipate that the mathematics will be simplified if each factor has the same dimensionality.
- Substitute this into the Laplace equation:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r\left[\frac{R(r)}{r}P(\theta)Q(\phi)\right]\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\left[\frac{R(r)}{r}P(\theta)Q(\phi)\right]\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\left[\frac{R(r)}{r}P(\theta)Q(\phi)\right]=0$$

$$P(\theta)Q(\phi)\frac{1}{r}\frac{\partial^2 R(r)}{\partial r^2}+\frac{R(r)}{r}Q(\phi)\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial P(\theta)}{\partial\theta}\right)+\frac{R(r)}{r}P(\theta)\frac{1}{r^2\sin^2\theta}\frac{\partial^2 Q(\phi)}{\partial\phi^2}=0$$

- This equation is complex enough that we can not make each term independent all at once. First, get Q in a form to show it is independent by multiplying by $r^3\sin^2\theta/R(r)P(\theta)Q(\phi)$:

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)+\frac{1}{Q(\phi)}\frac{d^2Q(\phi)}{d\phi^2}=0$$

- Here the partial derivatives have become total derivatives because the functions they operate on are now functions of only one variable.
- The last term is now independent of ρ and θ, and must hold for all ρ and θ, so that it must equal a constant:

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)-m^2=0 \quad \text{and} \quad -m^2=\frac{1}{Q(\phi)}\frac{d^2Q(\phi)}{d\phi^2}$$

- We can solve the second equation. First put it in a more intuitive form:

$$\frac{d^2Q(\phi)}{d\phi^2}=-m^2Q(\phi)$$

- Now the general solution is clearly:

$\boxed{Q(\phi)=A_m e^{im\phi}+B_m e^{-im\phi}}$ if $m \neq 0$ and $\boxed{Q(\phi)=A_{m=0}+B_{m=0}\phi}$ if $m = 0$.

- The constant m is in general not necessarily an integer. If the region of interest includes the full azimuthal sweep of values, then m must be an integer to keep the solution single-valued and the case of $m = 0$ reduces to $Q(\varphi) = A_{m=0}$. From here on, we are dealing with this special case, which is still quite general.

- We now turn to the rest of the equation:

$$\frac{r^2\sin^2\theta}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{\sin\theta}{P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)-m^2=0$$

- Divide each side by $\sin^2\theta$:

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2}+\frac{1}{\sin\theta\, P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=0$$

- The first term and the last terms are independent and can be set to a constant:

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2}-l(l+1)=0 \quad \text{where} \quad -l(l+1)=\frac{1}{\sin\theta\, P(\theta)}\frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)-\frac{m^2}{\sin^2\theta}$$

- Put both each equations in more intuitive forms :

$$\frac{d^2R(r)}{dr^2}=l(l+1)\frac{R(r)}{r^2} \quad \text{and} \quad \frac{d}{d\theta}\left(\sin\theta\frac{dP(\theta)}{d\theta}\right)+\left[l(l+1)-\frac{m^2}{\sin^2\theta}\right]\sin\theta\, P(\theta)=0$$

- Put the second equation in a simpler form using: $x=\cos\theta$ and $\frac{d}{d\theta}=-\sqrt{1-x^2}\frac{d}{dx}$:

$$\frac{d^2R(r)}{dr^2}=l(l+1)\frac{R(r)}{r^2} \quad \text{and} \quad \frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right]+\left[l(l+1)-\frac{m^2}{1-x^2}\right]P(x)=0$$

- The first equation can be solved by trying $R(r)=r^\alpha$

$$\alpha(\alpha-1)r^{\alpha-2}=l(l+1)r^{\alpha-2}$$

$$\alpha^2-\alpha-l(l+1)=0$$

$$\alpha=l+1 \quad , \quad \alpha=-l$$

- So that general solution is:

$\boxed{R(r)=A_l r^{l+1}+B_l r^{-l}}$ which is valid for both $l(l+1)\neq 0$ and $l(l+1)=0$

- Before solving further, we can already see that the general solution (if the whole azimuthal sweep is included) will have the form:

$$\Phi(r,\theta,\phi)=\sum_m \sum_l \left(A_l r^l+B_l r^{-l-1}\right)\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)P_l^m(\cos\theta)$$

where P_l^m at this point has not been solved yet, but will be the solution to the differential equation above involving theta.
-There are two separation constants *m* and *l*, and there are thus four cases that may need to be handled separately. A quick check of the above shows that it properly handles three cases. The ($m \neq 0$, $l(l+1) = 0$) case does not converge at $x = 1$ as we require and most be omitted.
- Solving the separated equation for the last spherical coordinate is involved and requires more detail as shown in the next section. We will take it in two steps. First we will solve it for the special case of $m = 0$ to get a feel for what the solution will be, then later we will move on and solve it for all *m*.

2. Ordinary Legendre Polynomials ($m = 0$)

- Solving for the $P(\theta)$ part of the potential is complex enough that we will take the special case $m = 0$ first and then treat the $m \neq 0$ case in a later lecture.

- To solve the equation $\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right]+l(l+1)P(x)=0$,

try a power series solution of the form $P(x)=\sum_{j=0}^{\infty} a_j x^{j+\alpha}$:

$$\sum_{j=0}^{\infty}(j+\alpha)(j+\alpha-1)a_j x^{j+\alpha-2}-(j+\alpha)(j+\alpha+1)a_j x^{j+\alpha}+l(l+1)a_j x^{j+\alpha}=0$$

$$\sum_{j=0}^{\infty}(j+\alpha)(j+\alpha-1)a_j x^{j+\alpha-2}-\sum_{j=0}^{\infty}[(j+\alpha)(j+\alpha+1)-l(l+1)]a_j x^{j+\alpha}=0$$

- Remove the first two powers of *x* and then combine the same powers in one sum:

$$(\alpha)(\alpha-1)a_0 x^{\alpha-2}+(1+\alpha)(\alpha)a_1 x^{\alpha-1}+\sum_{j=0}^{\infty}\left[(j+2+\alpha)(j+\alpha+1)a_{j+2}-[(j+\alpha)(j+\alpha+1)-l(l+1)]a_j\right]x^{j+\alpha}=0$$

- This must hold for all values of *x*, thus each power in the sum must be zero:

$(\alpha)(\alpha-1)a_0=0$, $(1+\alpha)(\alpha)a_1=0$, and

$(j+\alpha+2)(j+\alpha+1)a_{j+2}-[(j+\alpha)(j+\alpha+1)-l(l+1)]a_j=0$

- The third equation is solved to yield the recurrence relation, which gives us all remaining terms from the first one:

$$a_{j+2}=\frac{(j+\alpha)(j+\alpha+1)-l(l+1)}{(j+\alpha+2)(j+\alpha+1)}a_j$$

- The first two equations are redundant, so that we can permanently choose $a_1=0$. According to the recurrence relation above, these means that all $a_{odd}=0$. We can write the series now as:

$$P_l(x)=\sum_{j=0,\text{even}}^{\infty} a_j x^{j+\alpha}$$

- To satisfy the recurrence relations, we are left with two cases, $\alpha=0$ or $\alpha=1$.
- Depending on l, it is evident that $P_l(x)$ is either an odd series expansion or an even series.
- The solution only converges for $x=1$ if the series is finite. The series will only have a finite number of terms if the coefficient in the recurrence relation at some point equals zero:

$$\frac{(j_{max}+\alpha)(j_{max}+\alpha+1)-l(l+1)}{(j_{max}+\alpha+2)(j_{max}+\alpha+1)}=0$$

- We apply our two cases:

- If $\alpha=0$: $(j_{max})(j_{max}+1)-l(l+1)=0$ with the solution $j_{max}=l$, and plugging in:

$$P_l(x)=\sum_{j=0,\text{even}}^{l} a_j x^j$$

- We can rewrite the sum over even integers as a sum over all integers if we double the indices:

$$\boxed{P_l(x)=\sum_{j=0}^{l/2} a_{2j}x^{2j}} \quad \text{where} \quad \boxed{a_{j+2}=\frac{j(j+1)-l(l+1)}{(j+2)(j+1)}a_j} \quad \text{if } l \text{ is even}$$

- If $\alpha=1$: $(j_{max}+1)(j_{max}+2)-l(l+1)=0$ with the solution $j_{max}=l-1$, and plugging in:

$$P_l(x)=\sum_{j=0,\text{even}}^{l-1} a_j x^{j+1}$$

- Rewrite the sums over even integers as a sum over all integers by doubling the indices:

$$P_l(x)=\sum_{j=0}^{(l-1)/2} a_{2j}x^{2j+1}$$ where $$a_{j+2}=\frac{(j+1)(j+2)-l(l+1)}{(j+3)(j+2)}a_j$$ if l is odd

- We can explicitly show the solution for low values of l where the equations are simple:

$$P_0(x)=a_0$$

$$P_1(x)=a_0 x$$

$$P_2(x)=a_0+a_2x^2$$ where $a_2=-3a_0$ giving $P_2(x)=a_0(1-3x^2)$

$$P_3(x)=a_0x+a_2x^3$$ where $a_2=-\frac{5}{3}a_0$ giving $P_3(x)=a_0(x-\frac{5}{3}x^3)$

- The overall scale factor a_0 is arbitrary for each case and is conventionally set so that $$P_l(1)=1$$

$$P_0(x)=1$$

$$P_1(x)=x$$

$$P_2(x)=\frac{1}{2}(-1+3x^2)$$

$$P_3(x)=\frac{1}{2}(-3x+5x^3)$$

- We should remember that x is just a placeholder for cos θ in our analysis, so we really have:

$$P_0(\cos\theta)=1$$

$$P_1(\cos\theta)=\cos\theta$$

$$P_2(\cos\theta)=\frac{1}{2}(-1+3\cos^2\theta)$$

$$P_3(\cos\theta)=\frac{1}{2}(-3\cos\theta+5\cos^3\theta)$$

- Legendre Polynomials have the following mathematical properties (the derivations are left as exercises for the curious student).

- Odd or even symmetry about the origin: $P_l(-x)=(-1)^l P_l(x)$

- Rodrigues' Formula: $P_l(x)=\frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$

- Recurrence Relations: $P_{l+1}(x)=\frac{2l+1}{l+1}P_l(x)-\frac{l}{(l+1)}P_{l-1}(x)$

and: $P_l(x)=\frac{1}{2l+1}\frac{d}{dx}\left[P_{l+1}(x)-P_{l-1}(x)\right]$

- Note that the Legendgre polynomials are not defined for negative l, so that the above equations cannot be used to expand $P_0(x)$. When solving problems, this typically means that we have to solve the $l = 0$ term separately from the $l > 0$ terms.

- The orthogonality condition: $\int_{-1}^{1} P_{l'}(x)P_l(x)dx=\frac{2}{2l+1}\delta_{l'l}$

and in terms of the polar angle: $\int_{0}^{\pi} P_{l'}(\cos\theta)P_l(\cos\theta)\sin\theta d\theta=\frac{2}{2l+1}\delta_{l'l}$

- Note that the above orthogonality statement is only valid if the limits on the integral are zero and π. If you come across an integral with different limits, you cannot use the orthogonality statement to solve the integral.
- The Legendre Polynomials form a complete set of orthogonal functions on the interval (-1, 1), so any function $f(x)$ and be expanded in terms of Legendre Polynomials:

$$f(x)=\sum_{l=0}^{\infty} A_l P_l(x) \quad \text{where} \quad A_l=\frac{2l+1}{2}\int_{-1}^{1} f(x)P_l(x)dx$$

- The general solution to the Laplace Equation in spherical coordinates for the special case of ($m = 0$) has now been solved (when both poles require a finite solution):

$$\Phi(r,\theta,\phi)=\sum_{l=0}^{\infty}(A_l r^l+B_l r^{-l-1})P_l(\cos\theta) \quad \text{where } P_l(x) \text{ are the Legendre Polynomials.}$$

3. The Laplace Equation Solution for Problems with Azimuthal Symmetry

- If the condition $m = 0$ is met *and* if the region of valid solution includes the entire 2π radian sweep of ϕ , then the problem is said to have azimuthal symmetry. The solution above applies to any problem where the boundary conditions and do not depend on the azimuth angle, but are uniform in this direction.
- As an example, consider a sphere of radius a with the potential $V(\theta)$ on its surface and we wish to find the potential everywhere inside the sphere.
- Because the region of valid solution includes the origin, the constants $B_l = 0$ to keep the solution from blowing up, leading to:

$$\Phi(r,\theta)=\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

- Apply the boundary condition: $\Phi(r=a,\theta)=V(\theta)$

$$V(\theta)=\sum_{l=0}^{\infty} A_l a^l P_l(\cos\theta)$$

- Multiply both sides by $P_{l'}(\cos\theta)\sin\theta$ and integrate over theta from 0 to π:

$$\int_0^{\pi} V(\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta=\sum_{l=0}^{\infty} A_l a^l \int_0^{\pi} P_l(\cos\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta$$

- Now use the orthogonality condition: $\int_{-1}^{1} P_{l'}(x)P_l(x)dx=\frac{2}{2l+1}\delta_{l'l}$ where $x=\cos\theta$

$$\int_0^{\pi} V(\theta)P_{l'}(\cos\theta)\sin\theta\, d\theta=\sum_{l=0}^{\infty} A_l a^l \frac{2}{2l+1}\delta_{l'l}$$

$$A_l=\frac{2l+1}{2a^l}\int_{-1}^{1} V(\theta)P_l(\cos\theta)\sin\theta\, d\theta$$

- Thus the final solution is, in a form more intuitive:

$$\boxed{\Phi(r,\theta)=\sum_{l=0}^{\infty} A_l\left(\frac{r}{a}\right)^l P_l(\cos\theta)} \quad \text{where} \quad \boxed{A_l=\frac{2l+1}{2}\int_0^{\pi} V(\theta)P_l(\cos\theta)\sin\theta\, d\theta}$$

- Consider two hemispherical shells of radius *a* where the bottom half is held at zero and the top half is held at a fixed potential *V*.

$$A_l=\frac{2l+1}{2}V\int_0^{\pi/2} P_l(\cos\theta)\sin\theta\, d\theta$$

- Make the substitution $x=\cos\theta$:

$$A_l=\frac{2l+1}{2}V\int_0^{1} P_l(x)dx$$

- We have to be careful to do the $l = 0$ case separately, as is typical in this type of problem.
- For $l = 0$, we have:

$$A_0=\frac{1}{2}V\int_0^{1} P_0(x)dx$$

$$A_0=\frac{1}{2}V\int_0^{1} dx$$

$$\boxed{A_0 = \frac{1}{2} V}$$

- For $l > 0$, we use the relation $P_l(x) = \frac{1}{2l+1} \frac{d}{dx}\left[P_{l+1}(x) - P_{l-1}(x)\right]$ to find:

$$A_l = \frac{2l+1}{2} V \int_0^1 P_l(x) dx$$

$$A_l = \frac{1}{2} V \left[P_{l+1}(x) - P_{l-1}(x)\right]_0^1$$

$$A_l = \frac{1}{2} V \left[-P_{l+1}(0) + P_{l-1}(0)\right]$$

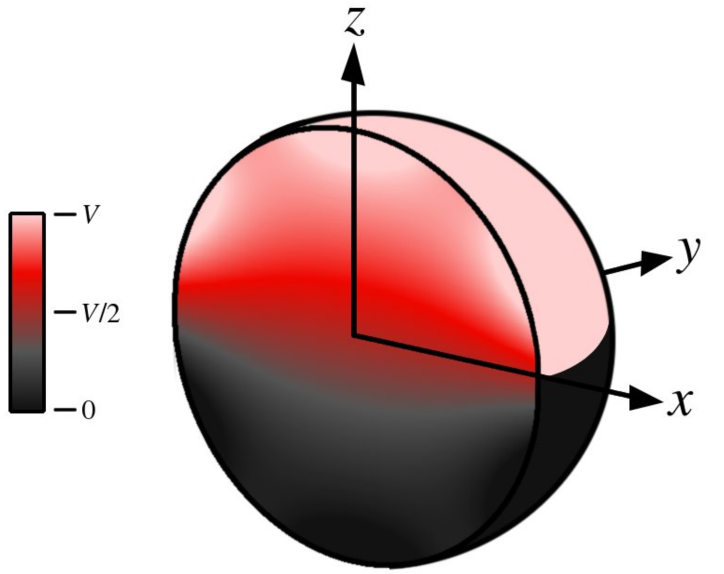
- So that the final solution is:

$$\Phi(r, \theta) = \frac{1}{2} V + \frac{1}{2} V \sum_{l=1}^{\infty} \left[-P_{l+1}(0) + P_{l-1}(0)\right] \left(\frac{r}{a}\right)^l P_l(\cos\theta)$$

- If r is much less than a, we can just keep the first few non-vanishing terms:

$$\begin{aligned}\Phi(r,\theta) \approx & \frac{1}{2} V \\ & + \frac{3}{4} V \left(\frac{r}{a}\right) \cos\theta \\ & - \frac{7}{32} V \left(\frac{r}{a}\right)^3 (5\cos^3\theta - 3\cos\theta) \\ & + \frac{11}{16^2} V \left(\frac{r}{a}\right)^5 (63\cos^5\theta - 70\cos^3\theta + 15\cos\theta)\end{aligned}$$

- Using graphing software, it is easy to plot these first four terms, as shown below.
- The plot demonstrates that even just keeping the first four terms gives a potential that meets the boundary conditions approximately, although it is obvious that accuracy is lost at certain points near the surface, when r is close to a.
- Many similar problems with azimuthal symmetry can be solved in the same way.



4. Unit Point Charge Potential Expansion in Legendre Polynomials.

- As we discovered when dealing with Green functions, the potential due to a point charge of unit magnitude ($q = 4\pi\varepsilon_0$) is a very useful building block. If we expand it in Legendre polynomials, we may be able to use it in these types of problems.
- Consider a unit charge placed on the z-axis. The potential felt at $\mathbf{r}$ from the unit charge at $\mathbf{r}_0$ on the z-axis is:

$$\Phi(r,\theta)=\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\frac{1}{\sqrt{r^2+r_0^2-2\,r\,r_0\cos\theta}}$$

- The potential is obviously azimuthally symmetric and can thus be expressed in terms of Legendre polynomials:

$$\frac{1}{\sqrt{r^2+r_0^2-2\,r\,r_0\cos\theta}}=\sum_{l=0}^{\infty}(A_l r^l+B_l r^{-l-1})P_l(\cos\theta)$$

- To find the coefficients A_l and B_l, we note that this equation must hold for all θ, so we can simplify the problem by picking a certain θ. Picking $\theta = 0$ yields:

$$\frac{1}{\sqrt{r^2+r_0^2-2\,r\,r_0}}=\sum_{l=0}^{\infty}(A_l r^l+B_l r^{-l-1})$$

$$\frac{1}{|r-r_0|}=\sum_{l=0}^{\infty}(A_l r^l+B_l r^{-l-1})$$

- For $r<r_0$ (the observation point is closer to the origin than the charge):

$$\frac{1}{r_0-r}=\sum_{l=0}^{\infty}(A_l r^l+B_l r^{-l-1})$$

- Multiply by r_0:

$$\frac{1}{1-\frac{r}{r_0}}=\sum_{l=0}^{\infty}\left(A_l r_0 r^l+B_l r_0 r^{-l-1}\right)$$

- Expand the left side using the geometric series rule: $\frac{1}{1-s}=\sum_{l=0}^{\infty} s^l$. This expansion only converges if $s < 1$. That is why we formed the above equation in terms of r/r_0.

$$\sum_{l=0}^{\infty}\left(\frac{r}{r_0}\right)^l=\sum_{l=0}^{\infty}\left(A_l r_0 r^l+B_l r_0 r^{-l-1}\right)$$

- It is now obvious that $B_l = 0$ and every term must match, leaving:

$$\left(\frac{r}{r_0}\right)^l=A_l r_0 r^l$$

$$A_l=r_0^{-(l+1)}$$

- Plugging this in gives us the final expansion:

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}_0|}=\sum_{l=0}^{\infty}\frac{r^l}{r_0^{l+1}}P_l(\cos\theta)} \quad \text{if} \quad \boxed{r<r_0}$$

- For $r>r_0$ (the observation point is further away from the origin than the charge):

$$\frac{1}{r-r_0}=\sum_{l=0}^{\infty}(A_l r^l+B_l r^{-l-1})$$

- Multiply by r:

$$\frac{1}{1-\frac{r_0}{r}}=\sum_{l=0}^{\infty}(A_l r^{l+1}+B_l r^{-l})$$

- Again expand the left into a geometric series:

$$\sum_{l=0}^{\infty}\left(\frac{r_0}{r}\right)^l=\sum_{l=0}^{\infty}(A_l r^{l+1}+B_l r^{-l})$$

- Now $A_l = 0$ and every term must match:

$$\left(\frac{r_0}{r}\right)^l=B_l r^{-l}$$

$B_l = r_0^l$

- Plugging this in gives us the final expansion:

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \sum_{l=0}^{\infty} \frac{r_0^l}{r^{l+1}} P_l(\cos\theta)} \quad \text{if} \quad \boxed{r > r_0}$$

- Strictly speaking, the point charge expansion shown above in boxes is only valid for a point charge on the *positive z* axis. But we can get an expression that is valid for a point charge on the *negative z* axis, if we substitute θ with (π – θ) because of the symmetry:

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \sum_{l=0}^{\infty} \frac{r^l}{r_0^{l+1}} P_l(\cos(\theta-\pi)) \quad \text{if} \quad r < r_0 \quad \text{and the point charge is on negative } z \text{ axis}$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \sum_{l=0}^{\infty} \frac{r^l}{r_0^{l+1}} P_l(-\cos\theta) \quad \text{if} \quad r < r_0 \quad \text{and the point charge is on negative } z \text{ axis}$$

- Using the property $P_l(-x) = (-1)^l P_l(x)$, we finally have

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \sum_{l=0}^{\infty} \frac{r^l}{r_0^{l+1}} (-1)^l P_l(\cos\theta)} \quad \text{if} \quad \boxed{r < r_0} \quad \text{and the point charge is on negative } z \text{ axis}$$

- Similarly, we can show:

$$\boxed{\frac{1}{|\mathbf{r}-\mathbf{r}_0|} = \sum_{l=0}^{\infty} \frac{r_0^l}{r^{l+1}} (-1)^l P_l(\cos\theta)} \quad \text{if} \quad \boxed{r > r_0} \quad \text{and the point charge is on negative } z \text{ axis}$$

- As an example of the usefulness of this expansion, consider a dipole where a charge *q* is located at *z* = *a* and -*q* is located at *z* = -*a*.
- The potential created by this dipole is:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}-a(\hat{\mathbf{z}})|} - \frac{q}{4\pi\epsilon_0} \frac{1}{|\mathbf{r}-a(-\hat{\mathbf{z}})|}$$

- For points far away from the origin and the charges:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} P_l(\cos\theta) - \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{a^l}{r^{l+1}} (-1)^l P_l(\cos\theta)$$

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \left(1-(-1)^l\right) \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

$$\Phi(\mathbf{r})=\frac{2q}{4\pi\epsilon_0}\sum_{l=0}^{\infty}\frac{a^{2l+1}}{r^{2l+2}}P_{2l+1}(\cos\theta)$$

- For very large *r* (or very small *a*, remembering that a perfect dipole is when *a* approaches zero) the potential can be approximated as the first term in this series:

$$\Phi(\mathbf{r})\approx\frac{q}{4\pi\epsilon_0}\frac{2a}{r^2}\cos\theta$$

- If we define the dipole moment as the charge times the separation: $p=q(2a)$ then the potential of a dipole takes on the familiar form:

$$\Phi(\mathbf{r})\approx\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}$$

where the approximately equal sign becomes an exactly equals sign for perfect dipoles.

5. Fields in a Conical Hole

- The Legendre polynomials form part of the solution to the Laplace equation if there is azimuthal symmetry *and* if the region where we must have a valid solution includes both poles $x = \pm 1$ ($\theta = 0, \pi$).
- What if there is azimuthal symmetry but the region of interest includes only the north pole, $x = 1$?
- Consider a conical hole in a conductor that has an angle β relative to its axis.
- In spherical coordinates, the boundary conditions are:

$$\Phi(r=0)=\text{finite},\ \Phi(\theta=0)=\text{finite},\ \Phi(\theta=\beta)=0,\ \Phi(0)=\Phi(2\pi)$$

- The boundary condition outside the hole is unknown so we cannot find a unique solution, but we can get a solution that is specific enough to describe the basic behavior of the fields in a conical hole.
- All of the pieces of the general $m = 0$ solution derived in the previous sections still hold except that $P_l(x)$ are now not the Legendre polynomials but are something else. Let us find them and label them $P_\nu(x)$ to avoid confusion.
- The Legendre equation was found to be (where $x = \cos\theta$):

$$\frac{d}{dx}\left[(1-x^2)\frac{dP(x)}{dx}\right]+\nu(\nu+1)P(x)=0$$

- When both poles were included there was a symmetry which lead us expand in a series solution about the midpoint $x = 0$ ($\theta = \pi/2$).
- But now only one pole is included, so the symmetry is lost. It will be cleaner now to expand in a series solution about the pole $x = 1$ ($\theta = 0$).
- For this reason, we make a change of variables:

$$x'=\frac{1}{2}(1-x)\quad,\quad\frac{d}{dx'}=-2\frac{d}{dx}$$

- The Legendre equation becomes:

$$\frac{d}{dx'}\left[(x'-x'^2)\frac{d\,P(x)}{dx'}\right]+\nu(\nu+1)P(x)=0$$

- As done in the previous sections, try a series solution of the form:

$$P(x)=\sum_{j=0}^{\infty} a_j x'^{j+\alpha}$$

- Substituting this into the equation we find:

$$\frac{d}{dx'}\left[(\sum_{j=0}^{\infty} a_j(j+\alpha)x'^{j+\alpha}-\sum_{j=0}^{\infty} a_j(j+\alpha)x'^{j+\alpha+1})\right]+\nu(\nu+1)\sum_{j=0}^{\infty} a_j x'^{j+\alpha}=0$$

$$\sum_{j=0}^{\infty} a_j(j+\alpha)^2 x'^{j+\alpha-1}+\sum_{j=0}^{\infty}[-(j+\alpha)(j+\alpha+1)+\nu(\nu+1)]a_j x'^{j+\alpha}=0$$

$$a_0\alpha^2 x'^{\alpha-1}+\sum_{j=0}^{\infty}[a_{j+1}(j+1+\alpha)^2+a_j[-(j+\alpha)(j+\alpha+1)+\nu(\nu+1)]]x'^{j+\alpha}=0$$

- The set of functions in the series is orthogonal so each coefficient must vanish separately, leading to:

$$a_0\alpha^2=0 \text{ and } a_{j+1}=a_j\frac{(j+\alpha)(j+\alpha+1)-\nu(\nu+1)}{(j+1+\alpha)^2}$$

- If we choose $a_0 = 0$ we will have no series, so we are forced to identify $\alpha = 0$, which leads to:

$$a_{j+1}=a_j\frac{(j)(j+1)-\nu(\nu+1)}{(j+1)^2}$$

- As a normalization, we choose $a_0 = 1$
- The final solution becomes:

$$P(x')=\sum_{j=0}^{\infty} a_j x'^j$$

$$P(x)=\sum_{j=0}^{\infty} a_j\left[\frac{1}{2}(1-x)\right]^j$$

$$P(x)=a_0+a_1\left[\frac{1}{2}(1-x)\right]+a_2\left[\frac{1}{2}(1-x)\right]^2+\ldots$$

$$\boxed{P_\nu(x)=1+(-\nu(\nu+1))\left[\frac{1}{2}(1-x)\right]+\frac{-\nu(\nu+1)(2-\nu(\nu+1))}{4}\left[\frac{1}{2}(1-x)\right]^2+\ldots}$$

- Note that we have not applied any boundary conditions yet so this solution is very general.
- If we apply the boundary condition of finite potential at both poles as done in the previous section, this forces υ to equal a positive integer and this general solution reduces down to the ordinary Legendre polynomials as it should to match the results of the previous section.
- For υ not equal to a positive integer, we can get solutions to other cases.
- By applying azimuthal symmetry and the finite potential requirement at the origin, our potential solution takes the form:

$$\Phi(r,\theta,\phi)=\sum_{\nu=0}^{\infty} A_\nu r^\nu P_\nu(\cos\theta)$$

- Apply the boundary condition $\Phi(\theta=\beta)=0$:

$$P_\nu(\cos\beta)=0$$

- Only certain $\upsilon = \upsilon_k$ will make this equation hold true. Unfortunately, they have to be found numerically for a certain β. Once found, the solution becomes:

$$\boxed{\Phi(r,\theta,\phi)=\sum_{k=1}^{\infty} A_k r^{\nu_k} P_{\nu_k}(\cos\theta)} \quad \text{where} \quad \boxed{P_{\nu_k}(\cos\beta)=0}$$

and

$$\boxed{P_\nu(x)=1+(-\nu(\nu+1))\left[\frac{1}{2}(1-x)\right]+\frac{-\nu(\nu+1)(2-\nu(\nu+1))}{4}\left[\frac{1}{2}(1-x)\right]^2+\ldots}$$

- This is as far as we can get without knowing the fields outside the hole. But this is still enough to show general trends deep inside the hole.
- Deep within the hole, near $r = 0$, the first term in the series solution will dominate so the other terms can be dropped:

$$\Phi(r,\theta,\phi)=A r^{\nu_1} P_{\nu_1}(\cos\theta)$$

- Therefore all the electric field components and the surface charge density vary radially as:

$$r^{\nu_1-1}$$

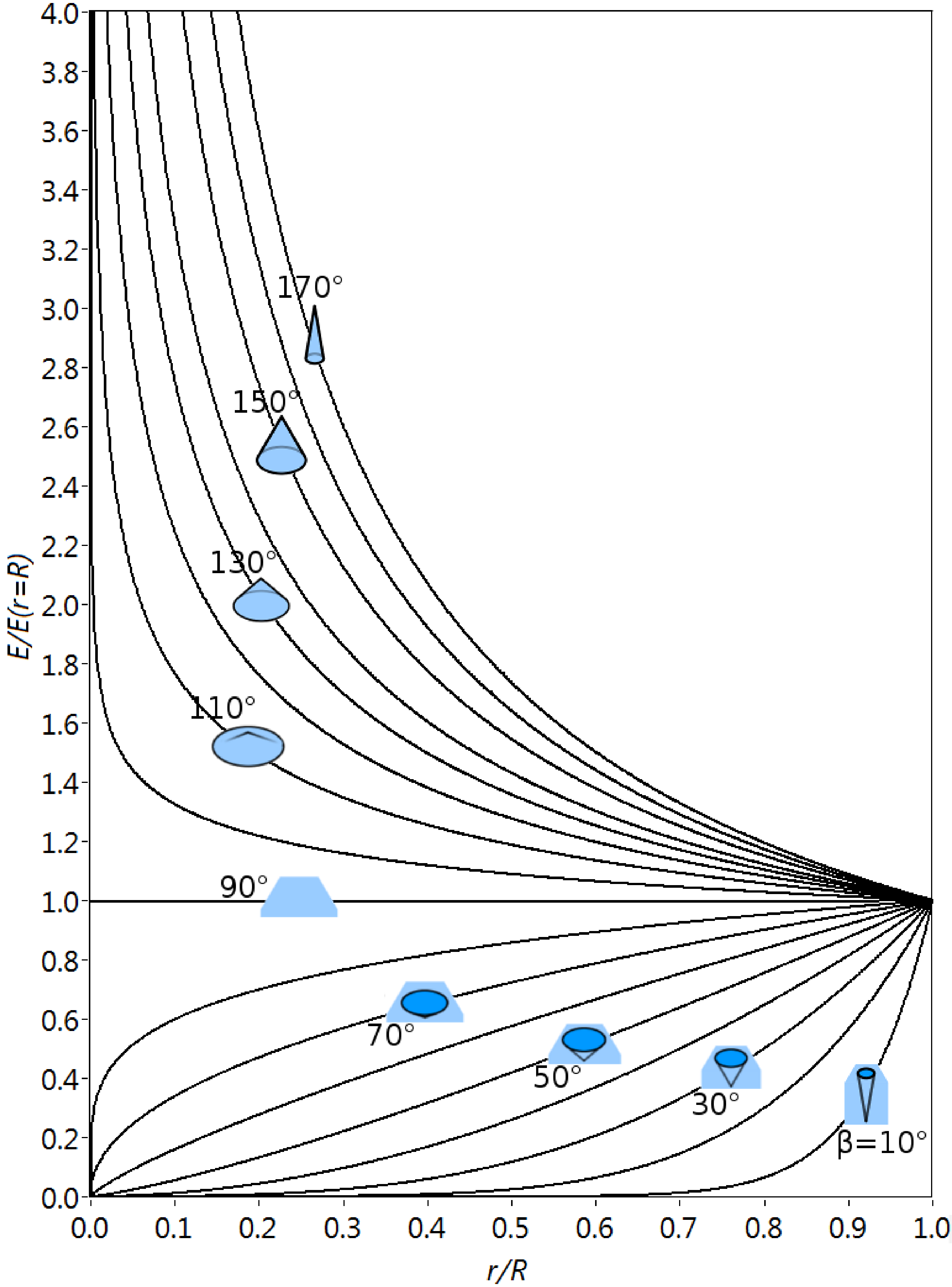
- For small β (conical holes), υ_1 approaches the value $\nu_1=\frac{2.405}{\beta}-\frac{1}{2}$

- For large β (conical points), υ_1 approaches the value $\nu_1=\left[2\ln\left(\frac{2}{\pi-\beta}\right)\right]^{-1}$

- Therefore the electric field and surface charge density varies as:

$$\boxed{r\,\hat{}\left[\frac{2.405}{\beta}-\frac{3}{2}\right]} \quad \text{for conical holes and} \quad \boxed{r\,\hat{}\left[\left[2\ln\left(\frac{2}{\pi-\beta}\right)\right]^{-1}-1\right]} \quad \text{for conical points}$$

- For instance the electric field inside a 45 degree conical hole is $E \propto r^{1.56}$.
- This means the field is steadily getting stronger the more we come out of the hole.
- For a narrow conical hole, such as with a 10 degree angle, we have $E \propto r^{12.3}$.
- This means that if we measure a field strength of 1 V/m near the mouth of the hole, than half way into the hole the field strength will be $2^{-12.3}$ V/m or 0.0002 V/m and three-quarters of the way into the hole the field strength will be 0.00000004 V/m.
- In general we may say that electric fields are very weak deep inside narrow holes
- For a 45 degree conical point (β = 135 degrees) we find $E \propto \frac{1}{r^{0.47}}$
- For a 10 degree conical point (β = 170 degrees) we find $E \propto \frac{1}{r^{0.79}}$
- For even sharper points, the field strength approaches the value $E \propto \frac{1}{r}$ so that if you go ten times closer to the point, the field strength gets ten times as strong.



Electric field strength near conical holes or conical points as a function of distance from the apex. All curves are normalized to have the same field strength at some fixed distance *R* from the apex.