Lecture 4 Notes, Electromagnetic Theory I

Dr. Christopher S. Baird
University of Massachusetts Lowell

1. Orthogonal Functions and Expansions

- In the interval (a, b) of the variable x, a set of real or complex functions $U_n(x)$ where n = 1, 2, ... are orthogonal if:

$$\int_a^b U_n^*(x)U_m(x)\,dx=0, \quad m\neq n$$

- When $m = n$, the integral is nonzero. The functions are orthonormal if normalized to one:

$$\int_a^b U_n^*(x)U_m(x)\,dx=\delta_{nm}$$

- An arbitrary, integrable function $f(x)$ can be expanded in a series of the orthonormal functions $U_n(x)$ according to:

$$f(x)=\sum_{n=1}^{N} a_n U_n(x)$$

- To find the expansion coefficients a_n, we multiply both sides by the function $U^*_M(x)$, integrate, and use the orthonormality property:

$$f(x)U_m^*(x)=\sum_{n=1}^{N} a_n U_n(x)U_m^*(x)$$

$$\int_a^b f(x)U_m^*(x)dx=\sum_{n=1}^{N} a_n \int_a^b U_n(x)U_m^*(x)\,dx$$

$$\int_a^b f(x)U_m^*(x)dx=\sum_{n=1}^{N} a_n \delta_{nm}$$

$$\int_a^b f(x)U_m^*(x)dx=a_m$$

- Interchange the arbitrary label m for n and get the final form:

$$\boxed{f(x)=\sum_{n=1}^{N} a_n U_n(x)} \text{ where } \boxed{a_n=\int_a^b f(x)U_n^*(x)\,dx}$$

Finite Series Expansion

- If the functions form a complete set, and all functions that are useful in physics do, then the series expansion becomes a more accurate representation of the function $f(x)$ as more terms in the series are kept. The most accurate is the infinite series:

$$\boxed{f(x)=\sum_{n=1}^{\infty} a_n U_n(x)} \text{ where } \boxed{a_n=\int_a^b f(x)\,U_n^*(x)\,dx} \qquad \textit{Infinite Series Expansion}$$

- If the interval (a, b) is expanded to be infinite, than the orthogonal functions become a continuum of functions, the index variable n becomes a continuous variable k, and the orthogonality condition becomes normalized to the Dirac delta function:

$$\int_{-\infty}^{\infty} U_k{}^*(x)\,U_{k'}(x)\,dx=\delta(k-k')$$

$$\boxed{f(x)=\int_{-\infty}^{\infty} A(k)\,U_k(x)\,dk} \text{ where } \boxed{A(k)=\int_{-\infty}^{\infty} f(x)\,U_n^*(x)\,dx} \quad \textit{Infinite Continuous Expansion}$$

2. Fourier Series

- The most commonly used orthogonal functions are sines and cosines, constituting Fourier series.
- Start with a general expansion in terms of sines and cosines over the interval (-a/2, a/2):

$$f(x)=\sum_{n=0}^{\infty} A_n\cos(k_n x)+B_n\sin(k_n x)$$

- In order for the series to be a valid representation of the function in the interval, the series must be periodic outside the interval, so that f(-a/2) = f(a/2). Using this requirement leads to:

$$\sum_{n=0}^{\infty} A_n(\cos(k_n(-a/2))-\cos(k_n a/2))+B_n(\sin(k_n(-a/2)-\sin(k_n a/2)))=0$$

- This must be true independent of A_n and B_n, so that the coefficients must be zero:

$$\sin(k_n(-a/2))-\sin(k_n a/2)=0$$

$$\sin(k_n a/2)=0 \quad \rightarrow \quad k_n a/2=n\pi$$

$$k_n=\frac{2\pi n}{a}$$

The series now becomes:

$$f(x)=\sum_{n=0}^{\infty} A_n\cos\left(\frac{2\pi n x}{a}\right)+B_n\sin\left(\frac{2\pi n x}{a}\right)$$

To find the coefficients, multiply both sides by $\cos\left(\frac{2\pi m x}{a}\right)$ where *m* is an integer and integrate over the interval (-*a*/2, *a*/2):

$$\int_{-a/2}^{a/2} dx\, f(x)\cos\left(\frac{2\pi m x}{a}\right)=\sum_{n=0}^{\infty} A_n \int_{-a/2}^{a/2} \cos\left(\frac{2\pi m x}{a}\right)\cos\left(\frac{2\pi n x}{a}\right)dx + B_n \int_{-a/2}^{a/2} \cos\left(\frac{2\pi m x}{a}\right)\sin\left(\frac{2\pi n x}{a}\right)dx$$

- The first integral on the right is zero, except when *m* = *n* due to orthogonality, and the second integral is always zero.

$$\int_{-a/2}^{a/2} dx\, f(x)\cos\left(\frac{2\pi m x}{a}\right)=A_m \int_{-a/2}^{a/2} \cos^2\left(\frac{2\pi m x}{a}\right)dx$$

- Use integration by parts to solve the integral on the right and finally after relabeling:

$$A_n=\frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\left(\frac{2\pi n x}{a}\right)dx$$

- The same approach is repeated, multiplying both sides by $\sin\left(\frac{2\pi m x}{a}\right)$ and integrating:

$$B_n=\frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\left(\frac{2\pi n x}{a}\right)dx$$

- In summary, any function on the interval (-*a*/2, *a*/2) can be expanded in a Fourier series:

$$\boxed{f(x)=\sum_{n=0}^{\infty} A_n\cos\left(\frac{2\pi n x}{a}\right)+B_n\sin\left(\frac{2\pi n x}{a}\right)}$$

where $\boxed{A_n=\frac{2}{a}\int_{-a/2}^{a/2} f(x)\cos\left(\frac{2\pi n x}{a}\right)dx}$ and $\boxed{B_n=\frac{2}{a}\int_{-a/2}^{a/2} f(x)\sin\left(\frac{2\pi n x}{a}\right)dx}$

- A more useful form of the Fourier series is in terms of complex exponentials:

$$\boxed{f(x)=\frac{1}{\sqrt{a}}\sum_{n=-\infty}^{\infty} A_n e^{i(2\pi n x/a)}} \text{ where } \boxed{A_n=\frac{1}{\sqrt{a}}\int_{-a/2}^{a/2} f(x')e^{-i(2\pi n x'/a)}dx'}$$

- Because the summation index now spans negative infinity to positive infinity instead of just zero to positive infinity, both terms e^{+ikx} and e^{-ikx} have been combined into one term. Sometimes it is more useful to leaves the terms separate.
- If the interval becomes infinite ($a \to \infty$), the series becomes an integral of a continuum of functions. There is now no restriction on k_n, so it is just *k:*

$$f(x)=\int_{-\infty}^{\infty} A(k)e^{ikx}dk$$

- Multiply both sides by $e^{-ik'x}$ and integrate:

$$\int_{-\infty}^{\infty} f(x)e^{-ik'x}dx = \int_{-\infty}^{\infty} A(k)\int_{-\infty}^{\infty} e^{-ik'x}e^{ikx}dx\,dk$$

- Using the orthogonality of complex exponentials: $\boxed{\int_{-\infty}^{\infty} e^{i(k-k')x}dx = 2\pi\delta(k-k')}$

$$A(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx$$

- More often, the constant A_n is redefined to $\frac{1}{\sqrt{2\pi}}A_n$ to make the equations symmetric:

$\boxed{f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{ikx}dk}$ where $\boxed{A(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dx}$ *Fourier Integral*

3. Separation of Variables: Laplace Equation in Rectangular Coordinates

- Often a differential equation can be broken into a set of independent equations.
- The Laplace equation $\nabla^2\Phi = 0$ is used when a charge-free region is bounded by a boundary where the potential is known. In rectangular coordinates:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0$$

- Try a solution of the form: $\Phi(x,y,z) = X(x)Y(y)Z(z)$

$$Y(y)Z(z)\frac{\partial^2 X(x)}{\partial x^2} + X(x)Z(z)\frac{\partial^2 Y(y)}{\partial y^2} + X(x)Y(y)\frac{\partial^2 Z(z)}{\partial z^2} = 0$$

- Divide each side by $X(x)Y(y)Z(z)$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0$$

- Total derivatives have replaced partial derivatives because the functions are of only one variable.
- This equation must hold for all possible values of the independent coordinates, therefore the terms must be independent. Each can be set to an arbitrary constant:

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -\alpha^2 \quad , \quad \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -\beta^2 \quad , \quad \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = \gamma^2 \quad \text{where} \quad \alpha^2 + \beta^2 = \gamma^2$$

- Simplify the equations:

$$\frac{d^2X(x)}{dx^2}=-\alpha^2X(x) \quad , \quad \frac{d^2Y(y)}{dy^2}=-\beta^2Y(y) \quad , \quad \frac{d^2Z(z)}{dz^2}=\gamma^2Z(z)$$

- And find the solutions:

$$X(x)=Ae^{i\alpha x}+Be^{-i\alpha x} \quad , \quad Y(y)=Ce^{i\beta y}+De^{-i\beta y} \quad , \quad Z(z)=Fe^{\gamma z}+Ge^{-\gamma z}$$

$$X(x)=A+Bx \text{ if } \alpha=0 \quad , \quad Y(y)=C+Dy \text{ if } \beta=0 \quad , \quad Z(z)=F+Gz \text{ if } \gamma=0$$

- The particular solution for non-zero constants is:

$$\Phi(x,y,z)=X(x)Y(y)Z(z)$$

$$\Phi(x,y,z)=(A_{\alpha\beta}e^{i\alpha x}+B_{\alpha\beta}e^{-i\alpha x})(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y})(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z}) \quad \text{if } \alpha\neq 0 \text{ and } \beta\neq 0$$

- The particular solutions for when the constants may be zero are:

$$\Phi(x,y,z)=(A_{0\beta}+B_{0\beta}x)(C_{0\beta}e^{i\beta y}+D_{0\beta}e^{-i\beta y})(F_{0\beta}e^{\beta z}+G_{0\beta}e^{-\beta z}) \quad \text{if } \alpha = 0 \text{ and } \beta\neq 0$$
$$\Phi(x,y,z)=(A_{\alpha 0}e^{i\alpha x}+B_{\alpha 0}e^{-i\alpha x})(C_{\alpha 0}+D_{\alpha 0}y)(F_{\alpha 0}e^{\alpha z}+G_{\alpha 0}e^{-\alpha z}) \quad \text{if } \alpha\neq 0 \text{ and } \beta = 0$$
$$\Phi(x,y,z)=(A_{00}+B_{00}x)(C_{00}+D_{00}y)(F_{\gamma=0}+G_{00}z) \quad \text{if } \alpha = 0 \text{ and } \beta = 0$$

- The general solution is the sum of all possible particular solutions:

$$\begin{aligned}\Phi(x,y,z)=&\sum_{\alpha\neq 0}\sum_{\beta\neq 0}(A_{\alpha\beta}e^{i\alpha x}+B_{\alpha\beta}e^{-i\alpha x})(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y})(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z})\\&+\sum_{\beta\neq 0}(A_{0\beta}+B_{0\beta}x)(C_{0\beta}e^{i\beta y}+D_{0\beta}e^{-i\beta y})(F_{0\beta}e^{\beta z}+G_{0\beta}e^{-\beta z})\\&+\sum_{\alpha\neq 0}(A_{\alpha 0}e^{i\alpha x}+B_{\alpha 0}e^{-i\alpha x})(C_{\alpha 0}+D_{\alpha 0}y)(F_{\alpha 0}e^{\alpha z}+G_{\alpha 0}e^{-\alpha z})\\&+(A_{00}+B_{00}x)(C_{00}+D_{00}y)(F_{00}+G_{00}z)\end{aligned}$$

- This is the most general solution possible. Most geometries are simple enough that most of these terms drop out.
- Let us now discuss differential equations in general.
- A derivative equation such as $\frac{dy}{dx}=2$ only specifies the solution up to an arbitrary constant. In this case $y = 2x + C$. We must have an additional piece of information, a boundary condition, to specify a unique solution to the derivative equation. In this case, if we know $y(1) = 4$, then we find the unique solution to be $y = 2x + 2$.
- A differential equation is just a combination of derivatives and functions.
- For every derivative in a differential equation, the solution will have one integration constant, and there must be one boundary condition to specify it.
- The Laplace equation in three-dimensions has a second-order derivative (which is really just a derivative applied twice) in each dimension, for a total of six derivatives. This means that for a particular solution, there will be six constants, and we need six boundary conditions to specify these constants. The six boundary conditions are just the value of the potential on the six sides of the box containing the volume of interest.
- Looking at the most common particular solution to Laplace's equation in three-dimensional

rectangular coordinates, it would seem that we have more than six constants:

$$\Phi(x,y,z)=(A_{\alpha\beta}e^{i\alpha x}+B_{\alpha\beta}e^{-i\alpha x})(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y})(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z})$$

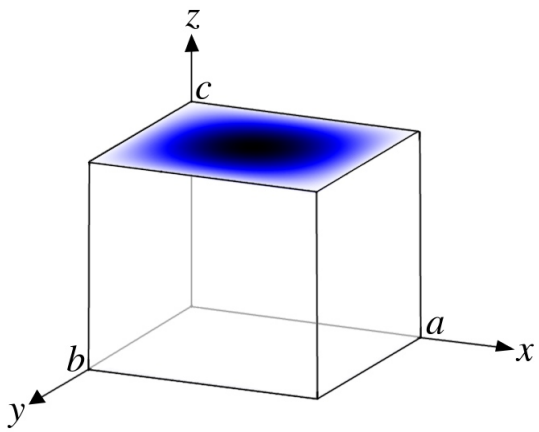
- But in reality, many of the constants can be combined and also γ is a function of the others constants:

$$\Phi(x,y,z)=A_{\alpha\beta}(e^{i\alpha x}+B_{\alpha\beta}e^{-i\alpha x})(e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y})(e^{\sqrt{\alpha^2+\beta^2}z}+G_{\alpha\beta}e^{-\sqrt{\alpha^2+\beta^2}z})$$

- It is a good practice when approaching a problem to write down the number of constants, identify them, and write *all* of the boundary conditions.

4. Example of Rectangular Boundary Conditions for Charge-Free Regions

- Consider for simplicity a box with one corner at the origin and the opposite corner at the point (a, b, c) in the (x, y, z) dimensions, where the potential is everywhere zero on the surface of the box except at the $z = c$ surface where:



$$\Phi(x,y,z=c)=\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$$

- The other five boundary conditions are the five other sides of the box held at zero.
- A careful analysis leads us to see that all the zero-potential sides force all particular solutions to disappear except the most common ones:

$$\Phi(x,y,z)=\sum_{\alpha\neq 0}\sum_{\beta\neq 0}(A_{\alpha\beta}e^{i\alpha x}+B_{\alpha\beta}e^{-i\alpha x})(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y})(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z})$$

- Now we apply all boundary conditions one by one:

$$\Phi(x=0,y,z)=0$$

$$0=\sum_{\alpha\neq 0}\sum_{\beta\neq 0}(A_{\alpha\beta}+B_{\alpha\beta})(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y})(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z})$$

- This is only true for all y and z if $A_{\alpha\beta}+B_{\alpha\beta}=0$, or $B_{\alpha\beta}=-A_{\alpha\beta}$.
- The solution is now:

$$\Phi(x,y,z)=\sum_{\alpha\neq 0}\sum_{\beta\neq 0}A_{\alpha\beta}(e^{i\alpha x}-e^{-i\alpha x})(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y})(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z})$$

$$\Phi(x,y,z)=\sum_{\alpha\neq 0}\sum_{\beta\neq 0}A_{\alpha\beta}\sin(\alpha x)(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y})(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z})$$

- Next apply the boundary condition $\Phi(x=a,y,z)=0$:

$$0=\sum_{\alpha\neq 0}\sum_{\beta\neq 0} A_{\alpha\beta}\sin(\alpha a)\left(C_{\alpha\beta}e^{i\beta y}+D_{\alpha\beta}e^{-i\beta y}\right)\left(F_{\alpha\beta}e^{\gamma z}+G_{\alpha\beta}e^{-\gamma z}\right)$$

- This is only true for all y and z if $\alpha a=n\pi$ where $n=0,1,2,...$ so that the solution becomes:

$$\Phi(x,y,z)=\sum_{n\neq 0}\sum_{\beta\neq 0} A_{n\beta}\sin\left(\frac{n\pi x}{a}\right)\left(C_{n\beta}e^{i\beta y}+D_{n\beta}e^{-i\beta y}\right)\left(F_{n\beta}e^{\gamma z}+G_{n\beta}e^{-\gamma z}\right)$$

- The exact same process occurs in the y dimension, yielding:

$$\Phi(x,y,z)=\sum_{n\neq 0}\sum_{m\neq 0} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left(F_{nm}e^{\gamma z}+G_{nm}e^{-\gamma z}\right) \quad \text{where } n,m=0,1,2,...$$

- With α and β determined, γ has been determined:

$$\gamma=\sqrt{\alpha^2+\beta^2} \quad \rightarrow \quad \gamma=\sqrt{\frac{n^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}} \quad \rightarrow \quad \gamma=\pi\sqrt{n^2/a^2+m^2/b^2}$$

- Next apply the boundary condition $\Phi(x,y,z=0)=0$:

$$0=\sum_{n\neq 0}\sum_{m\neq 0} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\left(F_{nm}+G_{nm}\right) \quad \text{and thus } G_{nm}=-F_{nm}\text{, yielding:}$$

$$\Phi(x,y,z)=\sum_{n\neq 0}\sum_{m\neq 0} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\pi\sqrt{n^2/a^2+m^2/b^2}\,z\right)$$

- Apply the last boundary condition: $\Phi(x,y,z=c)=\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$

$$\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)=\sum_{n\neq 0}\sum_{m\neq 0} A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\pi\sqrt{n^2/a^2+m^2/b^2}\,c\right)$$

- The only term of the series expansion that is needed to represent the left side of the equality is the $n = 1$, $m = 1$ term, reducing the equation to:

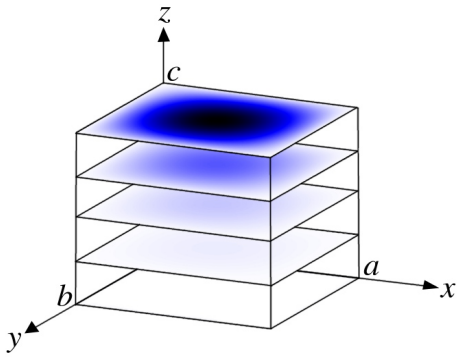
$$1=A_{1,1}\sinh\left(\pi\sqrt{1/a^2+1/b^2}\,c\right)$$

which uniquely determines the final coefficient:

$$A_{1,1}=\frac{1}{\sinh\left(\pi\sqrt{1/a^2+1/b^2}\,c\right)}$$

- The final solution to this example is:

$$\Phi(x,y,z)=\frac{\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)\sinh\left(\pi\sqrt{1/a^2+1/b^2}\,z\right)}{\sinh\left(\pi\sqrt{1/a^2+1/b^2}\,c\right)}$$



- We can make the case more general by supplying the arbitrary boundary condition:

$$\Phi(x,y,z=c)=V(x,y)$$

- Then the previous analysis still follows, except now the last boundary condition requires:

$$V(x,y)=\sum_{n\neq 0}\sum_{m\neq 0}A_{nm}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\pi\sqrt{n^2/a^2+m^2/b^2}\,c\right)$$

- The boundary value function V is being expanded in a Fourier series with coefficients:

$$A_{nm}\sinh\left(\pi\sqrt{n^2/a^2+m^2/b^2}\,c\right)$$

- As was done with the general Fourier series, we solve for the coefficients by multiplying both sides by sines of x and y and integrating, so that the orthogonality picks out coefficients:

$$A_{nm}\sinh\left(\pi\sqrt{n^2/a^2+m^2/b^2}\,c\right)=\frac{4}{a\,b}\int_0^a dx\int_0^b dy\,V(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

$$A_{nm}=\frac{4}{a\,b\sinh\left(\pi\sqrt{n^2/a^2+m^2/b^2}\,c\right)}\int_0^a dx\int_0^b dy\,V(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

- So that the general solution is:

$$\Phi(x,y,z)=\sum_{n\neq 0}\sum_{m\neq 0}A_{n,m}\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sinh\left(\pi\sqrt{n^2/a^2+m^2/b^2}\,z\right)$$

where $$A_{nm}=\frac{4}{a\,b\sinh\left(\pi\sqrt{n^2/a^2+m^2/b^2}\,c\right)}\int_0^a dx\int_0^b dy\,V(x,y)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)$$

5. The Laplace Equation in Polar Coordinates

- If the boundary conditions of a charge-free region exhibit *uniformity* in one dimension and a *circular* shape in the other two dimensions symmetry, it is much more natural mathematically to use polar coordinates rather than rectangular coordinates.
- If the boundary conditions are uniform in the *z*-dimension, the three-dimensional cylindrical-coordinates problem reduces to a two-dimensional polar-coordinates problem.
- Just like was done for rectangular coordinates, separation of variables and Fourier series can be used to solve the Laplace equation in polar coordinates.
- The two-dimensional Laplace equation in polar coordinates is given by:

$$\nabla^2 \Phi = 0 \quad \rightarrow \quad \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial \Phi}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 \Phi}{\partial \phi^2} = 0$$

- We use the separation of variables approach by trying a solution of the form:

$$\Phi(\rho, \phi) = R(\rho)\Psi(\phi)$$

- Substituting it in:

$$\Psi(\phi)\rho\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R(\rho)}{\partial \rho}\right) + R(\rho)\frac{1}{\rho^2}\frac{\partial^2 \Psi(\phi)}{\partial \phi^2} = 0$$

- Multiply by $\frac{\rho^2}{R(\rho)\Psi(\phi)}$ and bring one term to the right:

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R(\rho)}{\partial \rho}\right) = -\frac{1}{\Psi(\phi)}\frac{\partial^2 \Psi(\phi)}{\partial \phi^2}$$

- The two terms are separately functions of two independent variables and must hold for all values of the two independent variables, so they must be related by a constant, which we call v^2.

$$\frac{\rho}{R(\rho)}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R(\rho)}{\partial \rho}\right) = \nu^2 \quad \text{and} \quad \nu^2 = -\frac{1}{\Psi(\phi)}\frac{\partial^2 \Psi(\phi)}{\partial \phi^2}$$

- Put each in an instructive form:

$$\rho\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R(\rho)}{\partial \rho}\right) = \nu^2 R(\rho) \quad \text{and} \quad \frac{\partial^2 \Psi(\phi)}{\partial \phi^2} = -\nu^2 \Psi(\phi)$$

- The general solution for $\nu \neq 0$ is now apparent:

$$R(\rho) = a_\nu \rho^\nu + b_\nu \rho^{-\nu} \quad \text{and} \quad \Psi(\phi) = A_\nu e^{i\nu\phi} + B_\nu e^{-i\nu\phi}$$

- So that the general solution to the Laplace equation in polar coordinates for $\nu \neq 0$ is:

$$\Phi(\rho, \phi) = R(\rho)\Psi(\phi)$$

$$\Phi(\rho,\phi)=\sum_{\nu}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$

- If $\nu=0$, the differential equations become:

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R(\rho)}{\partial\rho}\right)=0 \quad \text{and} \quad \frac{\partial^2\Psi(\phi)}{\partial\phi^2}=0$$

with the general solutions for $\nu=0$:

$$R(\rho)=a_0+b_0\ln\rho \quad \text{and} \quad A_0+B_0\phi$$

- So that, finally, the most general solution becomes:

$$\boxed{\Phi(\rho,\phi)=(a_0+b_0\ln\rho)(A_0+B_0\phi)+\sum_{\nu\neq0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)}$$

- The coefficients are now determined by applying boundary conditions and recognizing the result as a Fourier series expansion.
- Because the *z* dimension is uniform and is therefore ignored, we have solved a second-order differential in two dimensions, so there should be four total integration constants and four boundary conditions. If we take the most common particular solution and combine constants, we find this to be the case:

$$\Phi_P(\rho,\phi)=a_\nu\left(\rho^\nu+b_\nu\rho^{-\nu}\right)\left(e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$

- Note that υ is only an integer if the region of interest where we are trying to solve for the potential includes all possible azimuthal angles, or in other words if there is no physical boundary condition at some fixed azimuthal angle. Otherwise, υ is not an integer. This means that the requirement that the potential be single-valued, $\Phi(\rho,\phi)=\Phi(\rho,\phi+2\pi)$, is only valid if the full sweep of possible azimuthal angles is included in the region of interest.

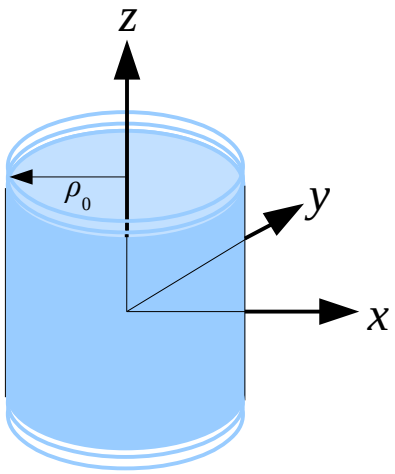
6. The Laplace Equation with *z*-Uniform Cylindrical-Shell Boundaries

- Consider a cylindrical shell of radius ρ_0 with a potential of $V(\varphi)$ on the shell. We are seeking the potential inside the shell.
- Because the region of interest involves the full sweep of φ, to keep φ single-valued we must require:

$$\Phi(\rho,\phi)=\Phi(\rho,\phi+2\pi)$$

- This condition is not true for geometries for which the region where a valid potential is required does not span all possible values of φ.
-In this geometry, the single-valued requirement leads to:

$$(a_0+b_0\ln\rho)(A_0+B_0\phi)+\sum_{\nu,\nu\neq0}\left(a_\nu\rho^\nu+b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$



$$=(a_0+b_0\ln\rho)(A_0+B_0(\phi+2\pi))+\sum_{\nu,\,\nu\neq 0}\left(a_\nu\rho^{\nu}+b_\nu\rho^{-\nu}\right)\left(A_\nu e^{i\nu(\phi+2\pi)}+B_\nu e^{-i\nu(\phi+2\pi)}\right)$$

- Because the variables are independent, every term in the series must match, leading to:

$B_0=0$

as well as:

$e^{i\nu\phi}=e^{i\nu(\phi+2\pi)}$ which is only possible if $\nu=m$ where m is an integer: $m=0,1,2,\ldots$

- The solution now becomes:

$$\Phi(\rho,\phi)=a_0+b_0\ln\rho+\sum_{m=1}^{\infty}\left(a_m\rho^{m}+b_m\rho^{-m}\right)\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)$$

- Because the region of interest includes the origin, the potential must be finite at the origin. Thus $b_m = 0$ and $b_0 = 0$, to keep those terms from blowing up at the origin. This reduces the solution to:

$\Phi(\rho,\phi)=\sum_{m=0}^{\infty}\rho^{m}\left(A_m e^{im\phi}+B_m e^{-im\phi}\right)$ where the a_0 term has now been included in the sum.

- The index m can be made to run from negative infinity to positive infinity, thus covering both terms:

$$\Phi(\rho,\phi)=\sum_{m=-\infty}^{\infty}A_m\rho^{|m|}e^{im\phi}$$

- Now apply the boundary condition: $\Phi(\rho=\rho_{0,}\phi)=V(\phi)$

$$V(\phi)=\sum_{m=-\infty}^{\infty}A_m\rho_0^{|m|}e^{im\phi}$$

- Multiply both side by a complex exponential, integrate, and use the orthogonality of exponentials to find:

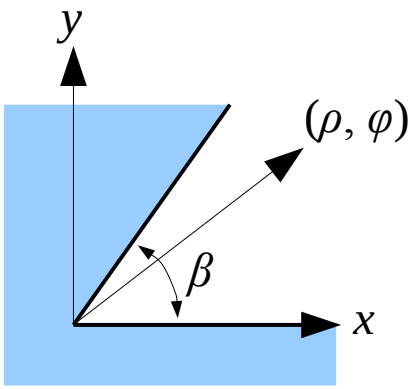
$$\Phi(\rho,\phi)=\sum_{m=-\infty}^{\infty}A_m\rho^{|m|}e^{im\phi}\quad\text{where}\quad A_m=\frac{1}{2\pi}\rho_0^{-|m|}\int_{-\pi}^{\pi}V(\phi')e^{-i(m\phi')}d\phi'$$

- Or, written in a more intuitive form:

$$\boxed{\Phi(\rho,\phi)=\sum_{m=-\infty}^{\infty}A_m\left(\frac{\rho}{\rho_0}\right)^{|m|}e^{im\phi}}\quad\text{where}\quad\boxed{A_m=\frac{1}{2\pi}\int_{-\pi}^{\pi}V(\phi')e^{-i(m\phi')}d\phi'}$$

- Although this example only considered one cylindrical shell, the same approach is used to solve for boundary conditions on multiple cylindrical shells.

7. The Laplace Equation with Intersecting Planes



- Next consider two planes that intersect, one on the *x*-axis, the other plane at an angle β from the first plane, and both uniform in the *z* dimension, with boundary conditions:

$$\Phi(\rho,\phi=0)=V \quad , \quad \Phi(\rho,\phi=\beta)=V \quad , \quad \Phi(\rho=\rho_{0,}\phi)=V_1(\phi)$$

- This is also applicable to the edge of a more complex problem, where the charges and other surfaces are far enough away that they only come into play in that they create the boundary condition at ρ_0, thus the Laplace equation still applies.
- As before, inclusion of the origin in the region of interest leads to $b_\upsilon = 0$, including $b_0 = 0$:

$$\Phi(\rho,\phi)=A_0+B_0\phi+\sum_{\nu,\nu\neq 0}\rho^{\nu}\left(A_\nu e^{i\nu\phi}+B_\nu e^{-i\nu\phi}\right)$$

- Apply the boundary condition $\Phi(\rho,\phi=0)=V$

$$V=A_0+\sum_{\nu,\nu\neq 0}\rho^{\nu}\left(A_\nu+B_\nu\right)$$

- Which is only possible if $A_0=V$, $A_\nu=-B_\nu$ so that the solution now becomes:

$$\Phi(\rho,\phi)=V+B_0\phi+\sum_{\nu,\nu\neq 0}B_\nu\rho^{\nu}\sin(\nu\phi)$$

- Apply the boundary condition $\Phi(\rho,\phi=\beta)=V$:

$$V=V+B_0\beta+\sum_{\nu,\nu\neq 0}B_\nu\rho^{\nu}\sin(\nu\beta)$$

$$0=B_0\beta+\sum_{\nu,\nu\neq 0}B_\nu\rho^{\nu}\sin(\nu\beta)$$

- This must be valid for all ρ, requiring $B_0=0$ and $\nu=m\pi/\beta$ where $m=0,1,2,\ldots$, yielding:

$$\Phi(\rho,\phi)=V+\sum_{m=1}^{\infty}B_m\rho^{m\pi/\beta}\sin(m\pi\phi/\beta)$$

- Note that ν is not equal to an integer in this case, but is equal to $\nu=m\pi/\beta$ because we have physical boundaries at certain azimuthal angles.
- Apply the final boundary condition, which is typically dependent on the charge far away:

$$\Phi(\rho=\rho_{0,}\phi)=V_1(\phi)$$

$$V_1(\phi)=V+\sum_{m=1}^{\infty}B_m\rho_0^{m\pi/\beta}\sin(m\pi\phi/\beta)$$

$$V_1(\phi)-V=\sum_{m=0}^{\infty} B_m \rho_0^{m\pi/\beta}\sin(m\pi\phi/\beta)$$

- Multiply both sides by $\sin(n\pi\phi/\beta)$ and integrate both sides from 0 to β:

$$\int_0^{\beta} d\phi\,(V_1(\phi)-V)\sin(n\pi\phi/\beta)=\sum_{m=0}^{\infty} B_m \rho_0^{m\pi/\beta}\int_0^{\beta} d\phi\sin(m\pi\phi/\beta)\sin(n\pi\phi/\beta)$$

- Due to orthogonality, the integral on the right is always zero, except when $m = n$:

$$\int_0^{\beta} d\phi\,(V_1(\phi)-V)\sin(n\pi\phi/\beta)=B_n \rho_0^{n\pi/\beta}\int_0^{\beta} d\phi\sin^2(n\pi\phi/\beta)$$

- The integral on the right evaluates to $\beta/2$:

$$B_n=\frac{2}{\beta}\rho_0^{-n\pi/\beta}\int_0^{\beta} d\phi\,(V_1(\phi)-V)\sin(n\pi\phi/\beta)$$

- The final solution then takes the form:

$$\boxed{\Phi(\rho,\phi)=V+\sum_{m=0}^{\infty} B_m \rho^{m\pi/\beta}\sin(m\pi\phi/\beta)} \quad , \quad \boxed{B_m=\frac{2}{\beta}\rho_0^{-m\pi/\beta}\int_0^{\beta} d\phi\,(V_1(\phi)-V)\sin(m\pi\phi/\beta)}$$

- Let us get a sense of the general behavior of the potential near the edge. In the series expansion, we keep only the fist non-zero term, $m = 1$:

$$\Phi(\rho,\phi)=V+B_1\rho^{\pi/\beta}\sin(\pi\phi/\beta) \quad \text{near } \rho=0$$

- The electric field, $\mathbf{E}=-\nabla\Phi$, is calculated in polar coordinates:

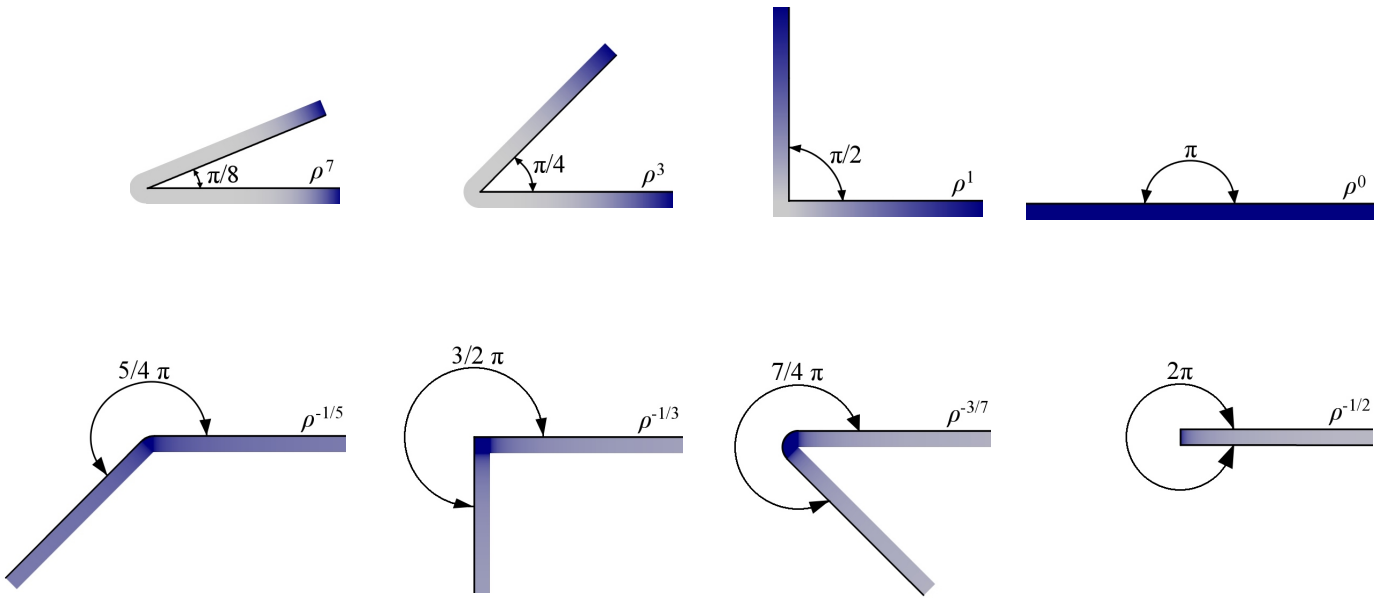
$$\mathbf{E}=-\left[\hat{\rho}\frac{\partial}{\partial\rho}+\hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\right]\left[V+B_1\rho^{\pi/\beta}\sin(\pi\phi/\beta)\right]$$

$$\boxed{\mathbf{E}=\frac{-\pi B_1}{\beta}\rho^{\pi/\beta-1}\left(\hat{\rho}\sin(\pi\phi/\beta)+\hat{\phi}\cos(\pi\phi/\beta)\right)}$$

- The surface charge densities at $\phi=0$ and $\phi=\beta$ are:

$$\sigma(\rho)=\left[\epsilon_0\mathbf{E}\cdot\mathbf{n}\right]_{\text{surface}} \quad \rightarrow \quad \sigma(\rho)=\epsilon_0\mathbf{E}(\rho,\phi=0)\cdot\hat{\phi}$$

$$\boxed{\sigma(\rho)=-\frac{\epsilon_0\pi B_1}{\beta}\rho^{\pi/\beta-1}}$$



Numerical Plots of surface charge densities for various angles, higher densities in darker blue.

8. Finite Element Analysis

- Often the Poisson equation cannot be solved analytically. Instead we must solve the problem numerically. Finite Element Analysis is one useful numerical method.
- This approach has three foundational ideas:
    1. Expand the electric potential's solution into a series sum over a set of simple orthogonal functions ϕ_i so that we can calculate the derivatives of the known functions instead of the unknown electric potential. Note that this is a simple one-dimensional set of functions, but that it spans two or three-dimensional space.
    2. Set up the problem so that it becomes a linear algebra problem, because computers can calculate this type of problem relatively quickly.
    3. Choose expansion functions that are localized to different points in space ("finite elements") so that the matrix ends up sparse and can be calculated much more quickly.

- Consider the Poisson equation in two dimensions valid in a region bounded by known Dirichlet boundary conditions:

$$\nabla^2 \psi = -g \text{ where } g = \rho/\epsilon_0$$

- Bring everything to the same side:

$$\nabla^2 \psi + g = 0$$

- Multiply everything by the test function so that we have it there to work with:

$$\phi_i \nabla^2 \psi + \phi_i g = 0$$

- Integrate this equation over the entire surface region S so that we have something that looks like Green's first identity:

$$\int_S [\phi_i \nabla^2 \psi + \phi_i g] da = 0$$

- Green's first identity for two-dimensions is:

$$\int_S (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d a = \oint_C \phi \frac{\partial \psi}{\partial n} d l$$

- We are assuming Dirichlet boundary conditions so that the derivative of the potential along the boundary (Neumann boundary conditions) can be set to zero making the right side of this equation go away.

$$\int_S (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d a = 0$$

$$\int_S \phi \nabla^2 \psi d a = -\int_S \nabla \phi \cdot \nabla \psi d a$$

- Apply this to the Poisson equation to obtain:

$$\int_S [-\nabla \phi_i \cdot \nabla \psi + \phi_i g] da = 0$$

$$\int_S \nabla \phi_i \cdot \nabla \psi da = \int_S \phi_i g da$$

- Now expand the electric potential into a sum of these expansion functions weighted by unknown coefficients A_j:

$$\psi = \sum_j A_j \phi_j$$

- Note that the index j represents the set of functions spanning both x and y dimensions.
- Apply this to the Poisson equation:

$$\int_S \nabla \phi_i \cdot \nabla \left[\sum_j A_j \phi_j \right] da = \int_S \phi_i g da$$

$$\sum_j A_j \left[\int_S \nabla \phi_i \cdot \nabla \phi_j da \right] = \int_S \phi_i g da$$

- At this point, the factor in square brackets depends only on our choice of expansion functions.
- If we choose a simple set of expansion functions, we can analytically precalculate the factors in the bracket and then just treat them as numbers computationally.
- The key to this approach is that we choose expansion functions that are only non-zero in a small finite region, so that the surface integral over these functions reduces to an integral over a small region. If this region is small enough, the source g is assumed to be constant across this region and can come out of the integral:

$$\sum_j A_j \left[\int_S \nabla \phi_i \cdot \nabla \phi_j da \right] = g_i \int_S \phi_i da$$

- Here g_i is the charge density at the location in space where function ϕ_i is non-zero.
- The right-hand side is just a set of numbers that can be calculate beforehand so that the problem has been reduced to a linear algebra problem where A_k are the unknowns.

$$\sum_j K_{ij} A_j = G_i \quad \text{where} \quad K_{ij} = \int_S \nabla \phi_i \cdot \nabla \phi_j da \quad \text{and} \quad G_i = g_i \int_S \phi_i da$$

- In matrix notation this becomes:

$$\mathrm{K}\,\mathbf{A} = \mathbf{G}$$

- The solution is:

$$\mathbf{A} = \mathrm{K}^{-1}\,\mathbf{G}$$

- The core computational task of solving the Poisson equation has been reduced to finding the inverse of a matrix, which computers can do very efficiently.
- Note that the potential is known along the boundaries and this must be included as well.
- The Finite Element method is therefore summarized as follows:

1. The human chooses a set of localized expansion functions ϕ_i
2. The human analytically calculates all $K_{ij} = \int_S \nabla \phi_i \cdot \nabla \phi_j da$ and $\int_S \phi_i da$
3. The computer finds the inverse K^{-1}
4. The computer calculates all $G_i = g_i \int_S \phi_i da$
5. The computer calculates $\mathbf{A} = \mathrm{K}^{-1}\,\mathbf{G}$
6. The computer calculates the final solution to the electric potential $\psi = \sum_j A_j \phi_j$