Lecture 3 Notes, Electromagnetic Theory I

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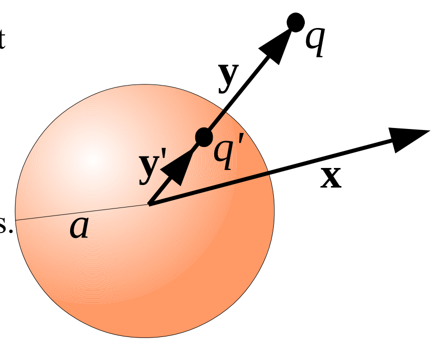
1. Method of Images

- Use the method of images when one or more point charges are in the presence of boundary surfaces with constant potentials across them.
- The method of images is very important because it can be used to find the Green function and then the Green function solution can be applied when the potential is not constant across the boundary.
- Replace the boundary surfaces with image charges external to the region of interest in locations recommended by symmetries.
- Make adjustable parameters out of the unknowns such as the image charge magnitude and location.
- Vary the adjustable parameters until the boundary condition at the surface is met.
- The solution to the original problem is the solution to the real charges and image charges.

2. Point Charge in the Presence of a Grounded Sphere



- Center the sphere of radius a at the origin, the real charge q at $\mathbf{y}$ and the observation point at $\mathbf{x}$.
- The grounded conducting sphere has zero potential at the surface: $\Phi(x=a)=0$
- If we sketch the field lines of the sphere and point, we see they look like the fields created by two charges and no surfaces. This gives us motivation that the method of images will work.
- Place the image charge q' inside the sphere at the point $\mathbf{y}'$.
- The potential due to the two charges using Coulomb's law is:

$$\Phi=\frac{1}{4\pi\epsilon_0}\frac{q}{|\mathbf{x}-\mathbf{y}|}+\frac{1}{4\pi\epsilon_0}\frac{q'}{|\mathbf{x}-\mathbf{y}'|}$$

- Split up vectors in terms of magnitude and directions.
Symmetry dictates that $\mathbf{y}$ and $\mathbf{y}'$ point in the same direction, so that $\mathbf{y}'=y'\hat{\mathbf{y}}$:

$$\Phi(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\frac{q}{|x\hat{\mathbf{x}}-y\hat{\mathbf{y}}|}+\frac{1}{4\pi\epsilon_0}\frac{q'}{|x\hat{\mathbf{x}}-y'\hat{\mathbf{y}}|}$$

- Apply the boundary condition:

$$\Phi(x=a)=0=\frac{1}{4\pi\epsilon_0}\frac{q}{|a\hat{\mathbf{x}}-y\hat{\mathbf{y}}|}+\frac{1}{4\pi\epsilon_0}\frac{q'}{|a\hat{\mathbf{x}}-y'\hat{\mathbf{y}}|} \quad \text{for all angles of the vector } \mathbf{x}$$

$$-\frac{q}{|a\hat{\mathbf{x}}-y\hat{\mathbf{y}}|}=\frac{q'}{|a\hat{\mathbf{x}}-y'\hat{\mathbf{y}}|}$$

- This must be true for all directions of the vector **x**, so we can pick out two different directions to derive two independent equations which we can then solve for our two unknowns.
- Let us first choose the vector **x** so that it points in the same direction as the vector **y**: $\hat{\mathbf{x}}=\hat{\mathbf{y}}$

$$-\frac{q}{|a\,\hat{\mathbf{y}}-y\,\hat{\mathbf{y}}|}=\frac{q'}{|a\,\hat{\mathbf{y}}-y'\,\hat{\mathbf{y}}|}$$

$$\boxed{-\frac{q}{y-a}=\frac{q'}{a-y'}}$$

- Now pick the vector **x** to point perpendicular to the vector **y** and expand the magnitudes according to $|\mathbf{r}_1-\mathbf{r}_2|=\sqrt{r_1^2+r_2^2-2\,\mathbf{r}_1\cdot\mathbf{r}_2}$:

$$-\frac{q}{\sqrt{a^2+y^2-2\,a\,y\,\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}}}=\frac{q'}{\sqrt{a^2+y'^2-2\,a\,y'\,\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}}}$$

$$-\frac{q}{\sqrt{a^2+y^2}}=\frac{q'}{\sqrt{a^2+y'^2}}$$

$$\boxed{\frac{q^2}{a^2+y^2}=\frac{q'^2}{a^2+y'^2}}$$

- We now have two independent equations (those in boxes above) in two unknowns q' and y' and can solve for the unknowns:

- Solve the first for q', $q'=\frac{q(y'-a)}{y-a}$ and substitute in the second:

$$\frac{q^2}{a^2+y^2}=\frac{\left(\frac{q(y'-a)}{y-a}\right)^2}{a^2+y'^2}$$

- After much algebra, get this in quadratic form and apply the quadratic equation:

$$y'^2+\left(\frac{a^2+y^2}{-y}\right)y'+a^2=0$$

$$\boxed{y'=\frac{a^2}{y}}$$

- Substitute back into the first:

$$q'=\frac{q\left(\left(\frac{a^2}{y}\right)-a\right)}{y-a}$$

$$q' = -\frac{a}{y} q$$

- Substituting back in, the solution for the potential becomes:

$$\Phi(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{\left|\frac{y}{a}\mathbf{x} - \frac{a}{y}\mathbf{y}\right|}\right]$$

- If the vector **y** is placed on the *z* axis and the vectors are expressed in spherical coordinates:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+y^2-2ry\cos\theta}} - \frac{1}{\sqrt{\frac{y^2}{a^2}r^2+a^2-2ry\cos\theta}}\right]$$

- We can use this to find the induced surface charge density on the sphere.

3. Surface Charge Density

- Start with the general form found previously:

$$\left[(\mathbf{E}_2-\mathbf{E}_1)\cdot\mathbf{n} = \frac{1}{\epsilon_0}\sigma\right]_{n=n_0}$$

- For a conductor, the electric field inside is everywhere zero, $\mathbf{E}_1 = 0$:

$$\left[\mathbf{E}_2\cdot\mathbf{n} = \frac{1}{\epsilon_0}\sigma\right]_{n=n_0}$$

- The electric field at the surface of a conductor is always parallel to the normal vector, and the surface here is just the set of points where the radial spherical coordinate *r* equals the radius *a*.

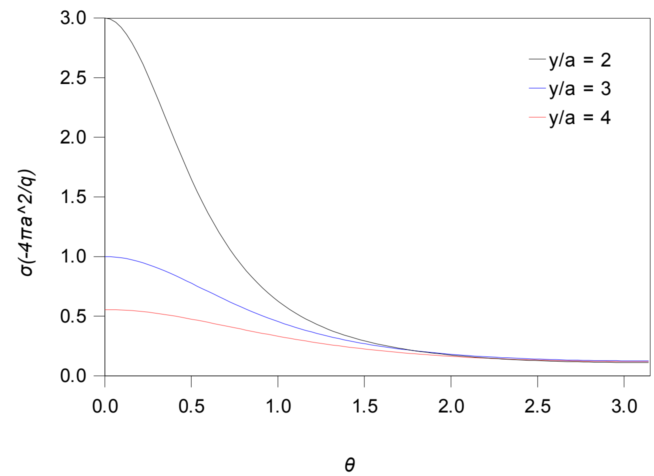
$$\left[E_2 = \frac{1}{\epsilon_0}\sigma\right]_{r=a}$$

- Expand the electric field in terms of the potential and solve for the charge density:

$$\sigma = \left[-\epsilon_0 \frac{d\Phi}{dr}\right]_{r=a}$$

- Now plug in the solution for the potential we obtained earlier, evaluate, and plot:

$$\sigma = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\frac{a^2}{y^2}}{\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\theta\right)^{3/2}}$$



4. Point Charge in the Presence of a Charged, Insulated, Conducting Sphere

- Consider a sphere with total charge Q in the presence of a point charge q.
- The situation is exactly the same as the grounded sphere, except that in addition to the induced charge, there is the rest of the total charge, $Q - q'$, which distributes itself uniformly over the sphere.
- There are thus three point charges, the real charge q at **y**, its corresponding image charge q' at **y**' with the same solutions as previously, and an image charge of magnitude $Q - q' = Q + qa/y$ at the origin.
- The potential can immediately be written down as the superposition of the potentials of the three charges:

$$\Phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{|\mathbf{x}-\mathbf{y}|} - \frac{q}{\left|\frac{y}{a}\mathbf{x}-\frac{a}{y}\mathbf{y}\right|} + \frac{Q+\frac{a}{y}q}{|\mathbf{x}|}\right]$$

- In spherical coordinates with the vector **y** on the z axis:

$$\Phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+y^2-2ry\cos\theta}} - \frac{1}{\sqrt{\frac{y^2}{a^2}r^2+a^2-2ry\cos\theta}} + \frac{\frac{Q}{q}+\frac{a}{y}}{r}\right]$$

- Using $\sigma = \left[-\epsilon_0\frac{d\Phi}{dr}\right]_{r=a}$ to find the surface charge density yields:

$$\sigma = -\frac{q}{4\pi a^2}\left(\frac{a}{y}\right)\frac{1-\frac{a^2}{y^2}}{\left(1+\frac{a^2}{y^2}-2\frac{a}{y}\cos\theta\right)^{3/2}} + \frac{1}{4\pi a^2}\left[Q+\frac{a}{y}q\right]$$

- The first term is just the induced surface charge found previously and the second term is just the remaining charge spread uniformly over the area of the sphere.

$$\sigma=\sigma_{\text{induced}}+\frac{[Q-q']}{A_{\text{sphere}}}$$

- Similarly, the force acting on charge q can immediately be written down as the superposition of the forces between them using Coulomb's law:

$$\mathbf{F}=\frac{q}{4\pi\epsilon_0}\sum_i q_i\frac{(\mathbf{x}-\mathbf{x}_i)}{|\mathbf{x}-\mathbf{x}_i|^3}$$

$$\mathbf{F}=\frac{q}{4\pi\epsilon_0}\left[q'\frac{(\mathbf{y}-\mathbf{y}')}{|\mathbf{y}-\mathbf{y}'|^3}+(Q-q')\frac{(\mathbf{y})}{|\mathbf{y}|^3}\right]$$

$$\boxed{\mathbf{F}=\frac{1}{4\pi\epsilon_0}\frac{q}{y^2}\left[Q-\frac{q a^3(2y^2-a^2)}{y(y^2-a^2)^2}\right]\hat{\mathbf{y}}}$$

- If instead of a charged, insulated conducting sphere, we have a sphere at fixed potential V, the image charge at the center is replaced by the charge (Va).

<u>5. Green Function for the Sphere</u>

- The potential due to a unit source and its image that satisfies homogeneous boundary conditions is the Green function for Dirichlet boundary conditions.
- We have already solved the potential for external to a grounded sphere and can write down the sphere's Green function immediately:

$$G(\mathbf{x},\mathbf{x}')=\frac{1}{|\mathbf{x}-\mathbf{x}'|}-\frac{1}{\left|\frac{x'}{a}\mathbf{x}-\frac{a}{x'}\mathbf{x}'\right|}$$

- In spherical coordinates, where both vectors are arbitrary now, not on the z axis:

$$\boxed{G(\mathbf{x},\mathbf{x}')=\frac{1}{\sqrt{x^2+x'^2-2xx'\cos\gamma}}-\frac{1}{\sqrt{\frac{x'^2}{a^2}x^2+a^2-2xx'\cos\gamma}}}\quad \textit{Sphere Green Function}$$

- Here, γ is the angle between the two vectors, so that in spherical coordinates:

$$\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$$

- Due to symmetry, this is the Green function for both cases, external to the sphere and internal to the sphere.
- Keep in mind that in this Green function expression, x is the magnitude of the observation point location vector **x**, and not the Cartesian coordinate. Likewise, x' is the magnitude of the source point location vector **x**', so that in spherical coordinates, $r = x$ and $r' = x'$. The variable a

is the radius of the sphere defining the boundary. If a specific problem uses the symbol *R*, *b*, etc. for the sphere radius, this should be used in place of the *a*.
- For the Green function to be useful, we must know it and its derivative normal to the surface, away from the volume of interest (the volume we are interested in is external, thus *n*' = -*x*'):

$$\left[\frac{dG}{dn'}\right]_{x'=a}=-\frac{(x^2-a^2)}{a(x^2+a^2-2xa\cos\gamma)^{3/2}}$$

- This is proportional to the surface charge density induced by a unit charge.
- For a problem that is internal to the sphere, the same expression for the normal derivative results except the overall sign is flipped because the normal points in the opposite direction.
- As an example, the potential outside a sphere with the potential specified on its surface but no charge involved ($\rho(\mathbf{x})=0$) can be found using the Green function method:

$$\Phi(\mathbf{x})=-\frac{1}{4\pi}\oint\left(\Phi\frac{dG_D}{dn'}\right)da'$$

- This is essentially the sum of all the infinitesimal surface charge patches induced by a unit charge weighted by the actual potential present at each surface patch. Plug in the Green function and write out explicitly in spherical coordinates:

$$\Phi(\mathbf{x})=\frac{1}{4\pi}\oint\left(\Phi\frac{(x^2-a^2)}{a(x^2+a^2-2xa\cos\gamma)^{3/2}}\right)da'$$

$$\boxed{\Phi(\mathbf{x})=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\left(\Phi(a,\theta',\phi')a\frac{(x^2-a^2)}{(x^2+a^2-2xa\cos\gamma)^{3/2}}\right)\sin\theta'\,d\theta'\,d\phi'}$$

6. Conducting Sphere with Hemispheres at Different Potentials

- Here is an example where the spherical Green function solution found above is useful.
- Consider a sphere of radius *a* centered on the origin split geometrically by the *z* = 0 plane so that the top half is held at +*V* potential and the bottom half is kept at -*V* potential.
- We can immediately use the result from above:

$$\Phi(\mathbf{x})=\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi/2}\left(V\,a\frac{(x^2-a^2)}{(x^2+a^2-2xa\cos\gamma)^{3/2}}\right)\sin\theta'\,d\theta'\,d\phi'+\frac{1}{4\pi}\int_0^{2\pi}\int_{\pi/2}^{\pi}\left((-V)a\frac{(x^2-a^2)}{(x^2+a^2-2xa\cos\gamma)^{3/2}}\right)\sin\theta'\,d\theta'\,d\phi'$$

where $\cos\gamma=\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')$

- Perform a change of variables on the second integral according to $\theta'\rightarrow\pi-\theta'$ and $\phi'\rightarrow\phi'+\pi$ so that $\cos\gamma\rightarrow-\cos\gamma$ and the integrals can be combined:

$$\Phi(\mathbf{x})=\frac{V\,a(x^2-a^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi/2}\left(\frac{1}{(x^2+a^2-2\,x\,a\cos\gamma)^{3/2}}-\frac{1}{(x^2+a^2+2\,x\,a\cos\gamma)^{3/2}}\right)\sin\theta'\,d\theta'\,d\phi'$$

- This can not be integrated in closed form in this case.
- However, we can investigate a special case of the potential anywhere on the z axis ($\theta = 0$):

$$\Phi(z)=\frac{V\,a(z^2-a^2)}{4\pi}\int_0^{2\pi}\int_0^{\pi/2}\left(\frac{1}{(z^2+a^2-2\,z\,a\cos\theta')^{3/2}}-\frac{1}{(z^2+a^2+2\,z\,a\cos\theta')^{3/2}}\right)\sin\theta'\,d\theta'\,d\phi'$$

Substitute $u=\cos\theta', du=-\sin\theta'\,d\theta'$:

$$\Phi(z)=\frac{V\,a(z^2-a^2)}{4\pi}\,2\pi\int_0^1\left(\frac{1}{(z^2+a^2-2\,z\,a\,u)^{3/2}}-\frac{1}{(z^2+a^2+2\,z\,a\,u)^{3/2}}\right)du$$

$$\Phi(z)=V\left(1-\frac{(z^2-a^2)}{z\sqrt{z^2+a^2}}\right)$$

$$\Phi(z)=V\left(1+\frac{1-\frac{z^2}{a^2}}{\sqrt{\frac{z^4}{a^4}+\frac{z^2}{a^2}}}\right)$$