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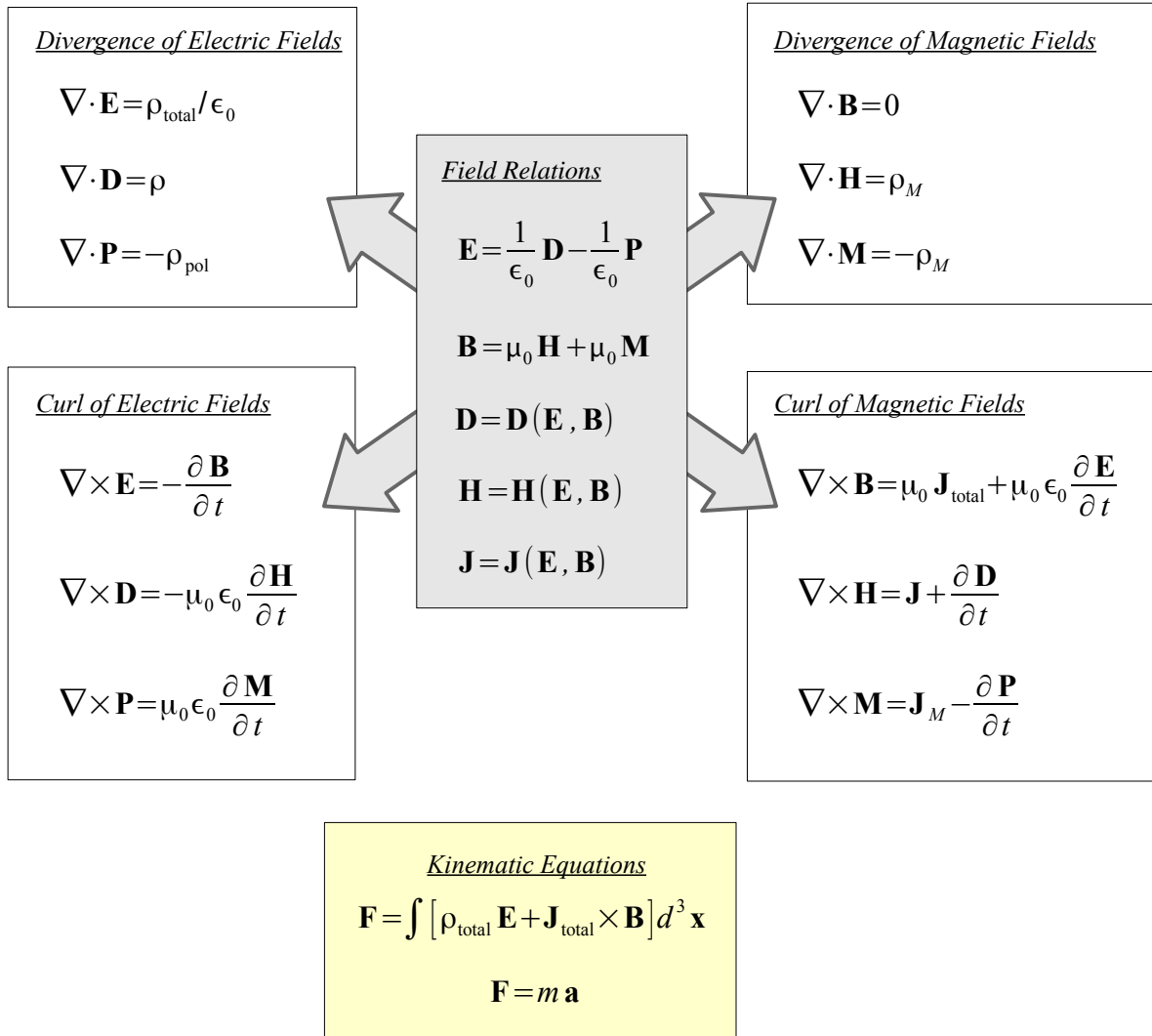
Lecture 14 Notes, Electromagnetic Theory I

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1. The Complete Equations of Classical Electrodynamics

- If we write down *all* the electrodynamic laws with materials included, we have:



- Once the constitutive equations are known; $\mathbf{D} = \mathbf{D}(\mathbf{E}, \mathbf{B})$ and $\mathbf{H} = \mathbf{H}(\mathbf{E}, \mathbf{B})$ (which describe the material's response); the field relations become complete. The completed field relations trivially link the \mathbf{E} , \mathbf{D} and \mathbf{P} fields as well as the \mathbf{B} , \mathbf{H} , and \mathbf{M} . As a result, when the completed field relations are applied, the twelve divergence and curl equations *collapse down to four independent equations*, one from each white box above. Which equation that we choose from each box above to be the independent one depends on personal taste and the specifics of the problem at hand. The two most popular choices are:

Collapsed Equations of Classical Electrodynamics In Terms of Total Fields:

Divergence of Electric Fields

$$\nabla \cdot \mathbf{E} = \rho_{\text{total}} / \epsilon_0$$

Divergence of Magnetic Fields

$$\nabla \cdot \mathbf{B} = 0$$

Curl of Electric Fields

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Curl of Magnetic Fields

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{total}} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Kinematic Equations

$$\mathbf{F} = \int [\rho_{\text{total}} \mathbf{E} + \mathbf{J}_{\text{total}} \times \mathbf{B}] d^3 \mathbf{x}$$

$$\mathbf{F} = m \mathbf{a}$$

Collapsed Equations of Classical Electrodynamics In Terms of Total and Partial Fields:

Divergence of Electric Fields

$$\nabla \cdot \mathbf{D} = \rho$$

Divergence of Magnetic Fields

$$\nabla \cdot \mathbf{B} = 0$$

Curl of Electric Fields

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

Curl of Magnetic Fields

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

Kinematic Equations

$$\mathbf{F} = \int [\rho_{\text{total}} \mathbf{E} + \mathbf{J}_{\text{total}} \times \mathbf{B}] d^3 \mathbf{x}$$

$$\mathbf{F} = m \mathbf{a}$$

2. Uniqueness in Maxwell's Equations

- Is there redundancy in Maxwell's equations? Don't Maxwell's equations over-specify the solution because there are eight scalar equations in the six unknowns, $E_x, E_y, E_z, B_x, B_y, B_z$?
- Let us answer this question by breaking equations into components.
- Helmholtz's decomposition theorem states that any well-behaved vector field \mathbf{A} can be decomposed into a transverse (i.e. solenoidal, curling, non-diverging) part \mathbf{A}_t and a longitudinal (i.e. diverging, irrotational, non-curling) part \mathbf{A}_l . The words "transverse" and "longitudinal" do not refer to the vector directionality of the components of \mathbf{A} , rather they specify the differential behavior of the components.

$$\mathbf{A} = \mathbf{A}_t + \mathbf{A}_l \quad \text{where} \quad \nabla \times \mathbf{A}_l = 0 \quad \text{and} \quad \nabla \cdot \mathbf{A}_t = 0$$

- If we expand the electric and magnetic field in Maxwell's equations into transverse and longitudinal parts we end up with:

$$(1) \quad \nabla \cdot \mathbf{E}_l = \frac{\rho}{\epsilon_0} \qquad (2) \quad \nabla \cdot \mathbf{B}_l = 0$$

$$(3) \quad \nabla \times \mathbf{E}_t = -\frac{\partial \mathbf{B}_t}{\partial t} - \frac{\partial \mathbf{B}_l}{\partial t} \qquad (4) \quad \nabla \times \mathbf{B}_t = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}_t}{\partial t} + \frac{1}{c^2} \frac{\partial \mathbf{E}_l}{\partial t}$$

- Taking the divergence of equations (3) and (4) and using the continuity equation, $\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$, we end up with modified forms for Faraday's law and the Ampere-Maxwell law:

$$(3') \quad \frac{\partial}{\partial t} (\nabla \cdot \mathbf{B}_l) = 0 \qquad (4') \quad \frac{\partial}{\partial t} \left(\nabla \cdot \mathbf{E}_l - \frac{\rho}{\epsilon_0} \right) = 0$$

- Equation (3'), which came from Faraday's law, may seem to be identical to equation (2), which is the no-magnetic-charge law. Similarly, equation (4'), which came from the Maxwell-Ampere law, may seem identical to equation (1), which is Coulomb's law.
- Coulomb's law (1) and the no-magnetic-charge law (2) therefore seem redundant and useless.
- Why are they included as part of Maxwell's equations?
- Coulomb's law and the no-magnetic-charge law are not redundant and the reason is the presence of the partial derivative operator with respect to time present in equations (3') and (4').
- Integrating away the time derivatives in (3') and (4') yields:

$$(3'') \quad \nabla \cdot \mathbf{B}_l = f(x, y, z) \qquad (4'') \quad \nabla \cdot \mathbf{E}_l = \frac{\rho}{\epsilon_0} + g(x, y, z)$$

- These equations show that Faraday's law and the Maxwell-Ampere law do not uniquely specify the longitudinal aspects of the fields, they only specify the time-evolution of the longitudinal aspects of the field, which turns out to be static (more correctly, statically linked, or instantaneously linked). We still need Coulomb's law and the no-magnetic charge law in order to get a unique solution.
- For this reason, Maxwell's equations are not redundant.
- Faraday's law and the Maxwell-Ampere law therefore dictate the *dynamical* behavior of all electric field and magnetic field components, including the longitudinal components, and

Coulomb's law and the no-magnetic-charge law serve only as initial conditions.

- Once we use Coulomb's law and the no-magnetic-charge law to properly establish initial conditions, then Faraday's law and the Maxwell-Ampere law alone dictate the fields the rest of the time.
- From a mathematical perspective, Maxwell's equations are not eight scalar equations in six unknowns, rather they are six scalar equations (Faraday's law and the Maxwell-Ampere law) in six unknowns plus two initial conditions.
- For this reason, Maxwell's equations are neither redundant nor over-specified.
- The presence of six equations in six unknowns means that we can in principle find a unique, consistent, solution to Maxwell's equations for a given set of properly-posed boundary/initial conditions.
- Note that transforming Maxwell's equations to wave-equation form does not reduce the number of dynamical equations, we still have six equations in six unknowns and we still need proper initial conditions. The advantage of the wave-equation form is solely that the field components are all completely decoupled.

3. Quasistatics

- The Helmholtz decomposition theorem (derived in the Supplemental Notes on the course website) tells us that if a vector field \mathbf{F} has a source to its divergence, $\psi = \nabla \cdot \mathbf{F}$, and a source to its curl, $\mathbf{U} = \nabla \times \mathbf{F}$, then the field can be expanded into integrals over these sources according to:

$$\mathbf{F}(\mathbf{x}) = \frac{1}{4\pi} \int \frac{\psi(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}' + \frac{1}{4\pi} \int \frac{\mathbf{U}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}' \quad \text{where } \psi = \nabla \cdot \mathbf{F} \text{ and } \mathbf{U} = \nabla \times \mathbf{F}$$

- This is a general mathematical result describing the properties of vector fields in general. Let us look at some applications in electromagnetics.

3.1 Electrostatics

- In electrostatics, the electric field \mathbf{E} is non-curling, $\mathbf{U} = 0$, so that the second term in the general expansion above goes away, and the source of the diverging electric field is the electric charge density, $\psi = \rho_{\text{total}}/\epsilon_0$. Plugging this in, the general expansion reduces to Coulomb's law:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_{\text{total}}(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}'$$

- Once experimental observations tell us that in electrostatics: (1) the electric field is non-curling and (2) the electric charge density is the source of the divergence, then mathematics gets us the rest of the way to Coulomb's law.

3.2 Magnetostatics

- In magnetostatics, the magnetic field \mathbf{B} is non-diverging, $\psi = 0$, so that the first term in the general vector expansion goes away. Also, the source of the curling magnetic field is the electric current density, $\mathbf{U} = \mu_0 \mathbf{J}_{\text{total}}$. Plugging this in, the expansion becomes the Biot-Savart law:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_{\text{total}}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}'$$

3.3 Magneto-quasi-statics

- In magneto-quasi-statics (MQS), all charges/currents are not static but travel or oscillate slowly enough compared to the speed of light that, to a good approximation, the magnetic field instantaneously tracks the currents that create them. This is true when magnetic fields dominate over electric fields. This is equivalent to setting $\frac{\partial \mathbf{E}}{\partial t} = 0$ in Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{total}}}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0$$

Magneto-quasi-static Equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{total}}$$

- In this realm, the magnetic field is solved exactly as in magnetostatics. Once solved, the magnetic field becomes a source for the electric field. In MQS, the electric and magnetic field are not completely coupled in an entire feedback loop. As a result, MQS describes electrostatic effects, magnetostatic effects, and induction effects, but does not include radiation effects.

- In magneto-quasi-statics, the source of the diverging electric field is the electric charge density, $\psi = \rho_{\text{total}}/\epsilon_0$, and the source of the curling electric field is the changing magnetic field, $\mathbf{U} = -\frac{\partial \mathbf{B}}{\partial t}$. The magnetic field is found by solving the magnetostatics problem. Plugging these sources into the general mathematical expansion of a vector field, we find:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_{\text{total}}(\mathbf{x}')(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d\mathbf{x}' - \frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{B}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d\mathbf{x}'$$

MQS Solution

$$\text{where } \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_{\text{total}}(\mathbf{x}') \times (\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3} d\mathbf{x}'$$

3.4 Electro-quasi-statics

In electro-quasi-statics (EQS), again all charges/currents travel or oscillate slowly, but here electric fields dominate over magnetic. The electric field instantaneously tracks the charges that create them. This is equivalent to setting $\frac{\partial \mathbf{B}}{\partial t} = 0$ in Maxwell's equations:

$$\nabla \cdot \mathbf{E} = \frac{\rho_{\text{total}}}{\epsilon_0}, \quad \nabla \cdot \mathbf{B} = 0$$

Electro-quasi-static Equations

$$\nabla \times \mathbf{E} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_{\text{total}} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

- In this realm, the electric field is solved exactly as in electrostatics and becomes a source for the magnetic field. Again, the electric and magnetic field are not completely coupled. As a result, EQS describes electrostatic effects, magnetostatic effects, and displacement current effects, but does not include radiation effects.

- In electro-quasi-statics, the magnetic field is not diverging, $\psi = 0$, and the source of the curling magnetic field is the total current as well as the changing electric field, $\mathbf{U} = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}$. The electric field is found by solving the electrostatics problem. Plugging these sources into the

general mathematical expansion of a vector field, we find:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_{\text{total}}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}' + \frac{1}{4\pi c^2} \frac{\partial}{\partial t} \int \frac{\mathbf{E}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}'$$

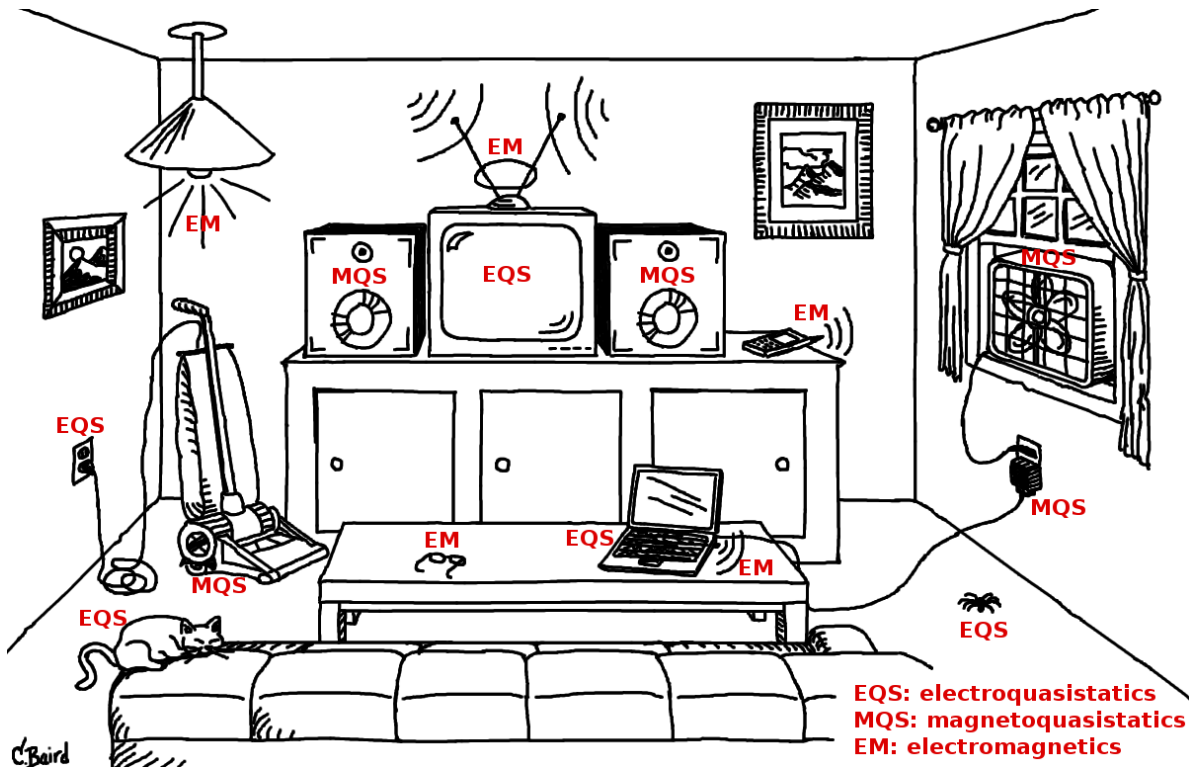
EQS Solution

$$\text{where } \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_{\text{total}}(\mathbf{x}')(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d\mathbf{x}'$$

3.5 Electrodynamics

- At high frequencies and high speeds, no approximations can be made and Maxwell's equations must be used in their full form. This means that the electric and magnetic field are completely coupled in a feedback loop leading to radiation. Because of this complete coupling, the sorting out of transverse and longitudinal sources/field components is too complicated to be presented here.

- The different devices in the home where EQS, MQS, and EM effects dominate are shown below.



4. Green Functions for the Wave Equation

- In the Lorenz gauge, the potentials obey wave equations. The wave equations can be thought of as the 4-dimensional equivalent of the Poisson equation, where the fourth dimension is time.
- Because of this equivalence, we can use the Green function method in the exact same way.
- Consider a region with no boundaries and no materials, only a free charge distribution.

$$\nabla^2 \Phi - \mu_0 \epsilon_0 \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0}$$

The corresponding Green function G must obey:

$$\left[\nabla^2 - \mu_0 \epsilon_0 \frac{\partial^2}{\partial t^2} \right] G = -4\pi \delta(\mathbf{x} - \mathbf{x}') \delta(t - t')$$

- The deltas only exist in one point in spacetime, so that everywhere else the solution to the source-less wave equation is just free traveling waves:

$$G \propto e^{\pm i\mathbf{k} \cdot \mathbf{x} - i\omega t}$$

- Here \mathbf{k} is the wave number and ω the angular frequency of the traveling wave solution where $k = \omega/c$ in free space.
- To accommodate the effects of the deltas, we shift this solution to the point charge. We also know that a point charge must have a $1/R$ dependence for potential-like entities, so that we have:

$$G = \frac{1}{|\mathbf{x} - \mathbf{x}'|} e^{\pm i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') - i\omega(t - t')}$$

- This is the same as the potential due to a spherical traveling wave propagating inward or outward from the point source.
- The general solution is the combination of all possible solutions. Because the wave number and frequency are not independent, we need only sum over one.

$$G = \frac{1}{2\pi} \int \frac{1}{|\mathbf{x} - \mathbf{x}'|} e^{\pm i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}') - i\omega(t - t')} d\omega$$

$$G = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \frac{1}{2\pi} \int e^{i\omega(t' - t \pm |\mathbf{x} - \mathbf{x}'|/c)} d\omega$$

- This is the general solution to the wave equation for the Green's function with a point source.
- The integral of a complex exponential is just a Dirac delta so that:

$$G = \frac{1}{|\mathbf{x} - \mathbf{x}'|} \delta(t' - t \pm |\mathbf{x} - \mathbf{x}'|/c)$$

- The solution to the wave equation for the potential using the Green function method is the integral of the sources weighted by the Green function (when no boundaries are present):

$$\Phi = \frac{1}{4\pi\epsilon_0} \int \int G(\mathbf{x}, \mathbf{x}', t, t') \rho(\mathbf{x}', t') d\mathbf{x}' dt'$$

- Plugging in the Green's function and evaluating the delta:

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{x}', t \pm |\mathbf{x} - \mathbf{x}'|/c)}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

- Amazingly, this is essentially the same equation we use in electrostatics, Coulomb's law, with one very important modification. The potential at time t is not that due to the charge distribution at time t , but that due to the charge distribution at an earlier time. (The + or – sign accounts for the two possibilities of an initial wave propagating in and being absorbed by the charges or an outgoing wave being created by the charges.

- The shift in time accounts for the finite time it takes for the fields created at the charge to reach out to the observation point. It is forced upon us by causality.

- This can be written more intuitively as:

$$\Phi(\mathbf{x}, t) = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho(\mathbf{x}', t)]_{\text{ret}}}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

where the “ret” subscript signifies we must use the retarded time, or the charge density at the earlier time.

- This is why when we look at stars hundreds of light years away, we are not seeing them as they are now but as they were hundreds of years ago when they created their electric and magnetic fields that are just now reaching us.

5. Introduction to Material Response in Electrodynamics

- Although the Maxwell equations in total-field form are the most straightforward, they are the least useful. The total-field forms require us to know the total charge density, $\rho_{\text{total}} = \rho + \rho_{\text{bound}}$, and the total current density, $\mathbf{J}_{\text{total}} = \mathbf{J} + \mathbf{J}_{\text{bound}}$. Usually we do not know the bound charges and currents, but must instead specify the material response a different way.

- The partial-fields form of the Maxwell equations is the most useful:

$$\nabla \cdot \mathbf{D} = \rho \quad , \quad \nabla \cdot \mathbf{B} = 0$$

Maxwell Equations for Partial Fields

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \quad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

- In this form, we do not need to know the bound charges and bound currents.

- But we do need to have some functional relationship between \mathbf{D} and \mathbf{E} as well as between \mathbf{H} and \mathbf{B} in order to solve the Maxwell equations. The functional relationships then contain the material effects instead of the bound charges and currents. Below are several cases.

Linear, Uniform, Isotropic, Dispersionless, Lossless Materials:

$$\mathbf{D} = \epsilon \mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu} \mathbf{B}$$

- Here ϵ and μ are independent of field strength (linear), independent of location (uniform), independent of direction (isotropic), independent of frequency (dispersionless), and are real numbers (lossless). Most materials can be approximated to behave like this. In such materials, the Maxwell equations reduce to:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon} \rho \quad , \quad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad , \quad \nabla \times \mathbf{B} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t}$$

Linear, Uniform, Isotropic, Dispersive, Lossy Materials

$$\mathbf{D} = \epsilon(\omega) \mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu(\omega)} \mathbf{B}$$

- Here $\epsilon(\omega)$ and $\mu(\omega)$ are the same as before, except they now *do* depend on the frequency ω of the fields (dispersive) and are complex numbered (lossy).
- A electromagnetic pulse that propagates can be thought of as a combination (a Fourier integral) of many harmonic waves, each with a distinct frequency ω . Each harmonic wave component has a different frequency and thus experiences a different material response due to the frequency-dependent permittivity and permeability. Some components will travel faster through the material. The net effect is that the pulse shape changes and the pulses disperses.
- When a material has an imaginary part to its permittivity and permeability, then the material absorbs energy from the electromagnetic wave.

Linear, Non-Uniform, Non-Isotropic, Dispersive, Lossy Materials.

$$\mathbf{D} = \epsilon(\omega, \mathbf{x}) \mathbf{E} \quad \text{and} \quad \mathbf{H} = \frac{1}{\mu(\omega, \mathbf{x})} \mathbf{B}$$

- Here the permittivity and permeability are now tensors so that each component of the fields are acted upon differently. They are also functions of space because the material is non-uniform.

Non-linear Materials

$$\mathbf{P} = f(\mathbf{E}, \mathbf{B}) \quad \text{and} \quad \mathbf{M} = g(\mathbf{E}, \mathbf{B})$$

- For non-linear materials, the material response can be quite complex and the physics must be handled on a case-by-case basis.
- For low frequencies, as an approximation, the electric and magnetic effects decouple:

$$\mathbf{P} = f(\mathbf{E}) \quad \text{and} \quad \mathbf{M} = g(\mathbf{E})$$

- The functional relationship can be expanded in a power series (shown in one dimension for simplicity)

$$\frac{1}{\epsilon_0} P = \chi^{(1)} E + \chi^{(2)} E^2 + \chi^{(3)} E^3 + \dots$$

- Often, many nonlinear effects can be explained quite accurately by only keeping the first few terms.

6. Conservation of Energy and the Poynting Vector

- In electrostatics, we found that the total work W required to assemble a static charge distribution is given by (this is equivalent to the potential energy stored in the electric fields):

$$W_{\text{electrostatics}} = \frac{1}{2} \int \mathbf{E} \cdot \mathbf{D} d^3 \mathbf{x}$$

- The integrand can then be thought of as a potential energy density w :

$$w_{\text{electrostatics}} = \frac{1}{2} \mathbf{E} \cdot \mathbf{D}$$

- Similarly, in magnetostatics we found that the potential energy W stored in the magnetic fields produced, is given by:

$$W_{\text{magnetostatics}} = \frac{1}{2} \int \mathbf{H} \cdot \mathbf{B} d^3 \mathbf{x} \quad \text{and the energy density is:}$$

$$w_{\text{magnetostatics}} = \frac{1}{2} \mathbf{H} \cdot \mathbf{B}$$

- The total energy stored if both charges and currents are present is then:

$$w = w_{\text{electrostatics}} + w_{\text{magnetostatics}}$$

$$w = \frac{1}{2} (\mathbf{E} \cdot \mathbf{D} + \mathbf{H} \cdot \mathbf{B})$$

- This equation is for static electromagnetic fields, but we should suspect that it applies to changing electromagnetic fields as well. Let us show this by considering the conservation of energy.

- The conservation of energy means that the rate at which electromagnetic energy W is lost out of the fields in a volume V should equal the rate at which kinetic energy W_k is gained by the charged objects interacting with the field, plus the energy W_c leaving V through its closed surface C :

$$-\frac{\partial W}{\partial t} = \frac{\partial W_k}{\partial t} + \frac{\partial W_c}{\partial t}$$

- Expand each energy variable in terms of an integral over the energy density:

$$-\int_V \frac{\partial w}{\partial t} d^3 \mathbf{x} = \int_V \frac{\partial w_k}{\partial t} d^3 \mathbf{x} + \oint_C \mathbf{n} \cdot \mathbf{S} da \quad \text{where } \mathbf{S} \text{ is the energy flow vector}$$

- Let us now try to get a more explicit form for the energy lost to kinetic energy of the charges w_k .

- If a force is applied to an object, it accelerates. The infinitesimal work done dW_K to accelerate the object is converted to the extra kinetic energy that the object gains.

$$d W_K = \mathbf{F} \cdot d \mathbf{x}$$

- Divide both sides by an infinitesimal period of time dt .

$$\frac{d W_K}{dt} = \mathbf{F} \cdot \frac{d \mathbf{x}}{dt}$$

$$\frac{d W_K}{dt} = \mathbf{F} \cdot \mathbf{v}$$

- If the object has a total charge q and the force is due to an electric field $\mathbf{F} = q \mathbf{E}$ (magnetic fields do no work) then:

$$\frac{d W_K}{dt} = (q \mathbf{v}) \cdot \mathbf{E}$$

- We can instead speak of a charge current density \mathbf{J} instead of the velocity of a single object. Then the total work is the spatial integral over the current density.

$$\frac{d W_K}{dt} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 \mathbf{x}$$

- The kinetic energy density w_k then becomes:

$$\frac{d w_k}{dt} = \mathbf{J} \cdot \mathbf{E}$$

-Plugging this in the equation above leads to:

$$\boxed{-\int_V \frac{\partial w}{\partial t} d^3 \mathbf{x} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 \mathbf{x} + \oint_C \mathbf{n} \cdot \mathbf{S} da}$$

Conservation of Energy

- We have built this equation up in a non-vigorous way using conceptual arguments. As such, we can make no statement about the electromagnetic energy density w held in the fields or the energy flow vector \mathbf{S} .

- Fortunately, the Maxwell equations already implicitly contain the conservation of energy equation.

- If we can get Maxwell equations into the form above, then we can make statements about w and \mathbf{S} .

- Let us start with the Ampere/Maxwell equation:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

- Dot each side by the electric field and integrate over the volume V :

$$\int_V (\nabla \times \mathbf{H}) \cdot \mathbf{E} d^3 \mathbf{x} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 \mathbf{x} + \int_V \left(\frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{E} d^3 \mathbf{x}$$

- Use the vector identity: $(\nabla \times \mathbf{H}) \cdot \mathbf{E} = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{H})$

$$\int_V \mathbf{H} \cdot (\nabla \times \mathbf{E}) d^3 \mathbf{x} - \int_V \left(\frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{E} d^3 \mathbf{x} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 \mathbf{x} + \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) d^3 \mathbf{x}$$

- Use Faraday's Law: $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

$$-\int_V \mathbf{H} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right) d^3 \mathbf{x} - \int_V \left(\frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{E} d^3 \mathbf{x} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 \mathbf{x} + \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) d^3 \mathbf{x}$$

$$-\int_V \left[\mathbf{H} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right) + \left(\frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathbf{E} \right] d^3 \mathbf{x} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 \mathbf{x} + \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) d^3 \mathbf{x}$$

- If the material is linear with negligible dispersion and losses, then $\left(\frac{\partial \mathbf{H}}{\partial t} \right) \cdot \mathbf{B} = \mathbf{H} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right)$

- This means that the product rule:

$$\frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{B}) = \mathbf{H} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right) + \left(\frac{\partial \mathbf{H}}{\partial t} \right) \cdot \mathbf{B} \quad \text{reduces to:} \quad \mathbf{H} \cdot \left(\frac{\partial \mathbf{B}}{\partial t} \right) = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{B})$$

and the same argument is used to show $\mathbf{E} \cdot \left(\frac{\partial \mathbf{D}}{\partial t} \right) = \frac{1}{2} \frac{\partial}{\partial t} (\mathbf{E} \cdot \mathbf{D})$

- We can use these relations to move the derivative out:

$$-\int_V \frac{\partial}{\partial t} \left[\frac{1}{2} (\mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D}) \right] d^3 \mathbf{x} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 \mathbf{x} + \int_V \nabla \cdot (\mathbf{E} \times \mathbf{H}) d^3 \mathbf{x}$$

- Use of the divergence theorem on the last term leads to:

$$-\int_V \frac{\partial}{\partial t} \left[\frac{1}{2} (\mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D}) \right] d^3 \mathbf{x} = \int_V \mathbf{J} \cdot \mathbf{E} d^3 \mathbf{x} + \oint_C \mathbf{n} \cdot (\mathbf{E} \times \mathbf{H}) d a$$

- We now compare this to the conceptual form of the conservation of energy (the boxed equation above). This equation *is* the conservation of energy equation if we recognize the

energy density w and energy density flow vector \mathbf{S} to be:

$$w = \frac{1}{2}(\mathbf{H} \cdot \mathbf{B} + \mathbf{E} \cdot \mathbf{D}) \quad , \quad \mathbf{S} = \mathbf{E} \times \mathbf{H}$$

- The energy density for nonstatic fields ends up the same as for the static fields.
- The energy density flow vector \mathbf{S} is called the Poynting vector.
- With use of the divergence theorem, the Poynting theorem (conservation of energy) can be put in differential form:

$$-\frac{\partial w}{\partial t} = \mathbf{J} \cdot \mathbf{E} + \nabla \cdot \mathbf{S}$$

Conservation of Energy for Linear, Loss-less Materials

7. Conservation Laws

- In a similar way, other conservation laws can be derived from Maxwell's equations.
- The following equations are implicitly contained in Maxwell's equations, the Lorenz force law, and Newton's law:

- Conservation of Charge: $\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}$

- Conservation of Linear Momentum: $\frac{d}{dt}(\mathbf{p}_{\text{mech}} + \mathbf{p}_{\text{field}}) = -\nabla \cdot \mathbf{T}$

where \mathbf{p}_{mech} is the mechanical momentum density of the currents,

the momentum density of the fields is: $\mathbf{p}_{\text{field}} = \frac{1}{c^2} \mathbf{E} \times \mathbf{H}$,

the momentum density flow tensor (the Maxwell stress tensor) is:

$$T_{ij} = \epsilon_0 E_i E_j + \epsilon_0 c^2 B_i B_j - \frac{1}{2} \epsilon_0 \delta_{ij} (E^2 + c^2 B^2) \quad ,$$

and the divergence operator is a second-rank tensor divergence.

- Conservation of Angular Momentum: $\frac{\partial}{\partial t}(\mathbf{r} \times \mathbf{p}_{\text{mech}} + \mathbf{r} \times \mathbf{p}_{\text{field}}) = -\nabla \cdot (\mathbf{r} \times \mathbf{T})$