Lecture 13 Notes, Electromagnetic Theory I

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1. Static Equations and Faraday's Law

- The two fundamental equations of electrostatics are shown below.

$$\nabla\cdot\mathbf{E}=\frac{\rho_{\text{total}}}{\epsilon_0}$$ *Coulomb's Law in Differential Form*

- Coulomb's law is the statement that electric charges create diverging electric fields.

$$\nabla\times\mathbf{E}=0$$ *Irrotational Electric Fields when Static*

- This means that if everything is static, then the electric fields have no curl.
- The two fundamental equations of magnetostatics are shown below:

$$\nabla\cdot\mathbf{B}=0$$ *No Magnetic Monopoles*

- Just as electric charges give rise to diverging electric fields, magnetic charges would give rise to diverging magnetic fields. But *there are no* magnetic charges (no magnetic monopoles). So there is no divergence to the magnetic fields. There other equation of magnetostatics is:

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}}$$ *Ampere's Law for Steady Currents*

- This equation states that steadily moving electric charges give rise to curling magnetic fields.
- These four equations completely specify all electromagnetic fields when everything is static in time. But what happens if something changes in time?
- Faraday was the first to show that these equations are not complete if we want to include time-varying effects. He showed that changing magnetic fields give rise to curling electric fields. The irrotational **E** field equation of electrostatics became Faraday's law in electrodynamics. The four equations now stood as:

$$\nabla\cdot\mathbf{E}=\frac{\rho_{\text{total}}}{\epsilon_0}$$ *Coulomb's Law in Differential Form*

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$ *Faraday's Law of Induction*

$$\nabla\cdot\mathbf{B}=0$$ *No Magnetic Monopoles*

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}}$$ *Ampere's Law for Steady Currents*

- But these four equations are now logically and mathematically inconsistent if we are no longer considering static situations.

- To show this, take the divergence of Ampere's law:

$$\nabla\cdot(\nabla\times\mathbf{B})=\mu_0\nabla\cdot\mathbf{J}_{\text{total}}$$

- Mathematically speaking, the divergence of the curl (shown on the left) is always zero, leading to:

$$0=\nabla\cdot\mathbf{J}_{\text{total}}$$

- This equation was fine for static situations, but for non-static situations, the continuity equation states:

$$-\frac{\partial\rho_{\text{total}}}{\partial t}=\nabla\cdot\mathbf{J}_{\text{total}}$$

- In non-static situations, the time-derivative of the charge density is non-zero, directly contradicting the equation above it.

2. The Maxwell Equations

- It took the genius of Maxwell to realize this problem and figure out how to fix it. For this accomplishment he is now honored with the distinction of having the final four equations named after him.
- Maxwell realized that to remove the contradiction, he could add an extra term to Ampere's law that would automatically make the continuity equation hold true. (Note: the continuity equation is more of a consistency check than a fundamental law about the fields. It should be contained automatically in the electromagnetic field equations.)
- Let us start with the continuity equation and work backwards to see what the more complete form of Ampere's law should look like.

$$-\frac{\partial\rho_{\text{total}}}{\partial t}=\nabla\cdot\mathbf{J}_{\text{total}}$$

- Plug in Coulomb's law into the left side of the continuity equation:

$$-\frac{\partial(\epsilon_0\nabla\cdot\mathbf{E})}{\partial t}=\nabla\cdot\mathbf{J}_{\text{total}}$$

- We can always add zero to an equation. Let us add the statement $\frac{1}{\mu_0}\nabla\cdot(\nabla\times\mathbf{B})=0$ so that we end up with Ampere's law:

$$-\frac{\partial(\epsilon_0\nabla\cdot\mathbf{E})}{\partial t}+\frac{1}{\mu_0}\nabla\cdot(\nabla\times\mathbf{B})=\nabla\cdot\mathbf{J}_{\text{total}}$$

- Integrate the divergence out of every term and rearrange:

$$\boxed{\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}}$$ *Ampere's Law with Maxwell's Correction*

- This matches the original Ampere's law, except there is an extra term now to make the continuity equation hold.
- With Ampere's law in complete form, and the other three equations still holding true, these four equations now describe all electrodynamics. They are known as the Maxwell equations:

$$\boxed{\begin{aligned} &\nabla\cdot\mathbf{E}=\frac{\rho_{\text{total}}}{\epsilon_0}\ , && \nabla\cdot\mathbf{B}=0 \\ &\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\ , && \nabla\times\mathbf{B}=\mu_0\,\mathbf{J}_{\text{total}}+\mu_0\,\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \end{aligned}}$$

Maxwell Equations for Total Fields

- Because Faraday's law says magnetic fields can create electric fields and Ampere's corrected law says that electric fields can create magnetic fields, there is a feedback process where they can create each other cyclically, independent of any charges, currents, or materials. This is the basis behind electromagnetic radiation.
- The Maxwell equations can be cast in a more useful, but less intuitive form, in terms of partial fields instead of total fields, and in terms of free currents/charges instead of total current/charges:

$$\boxed{\begin{aligned} &\nabla\cdot\mathbf{D}=\rho\ , && \nabla\cdot\mathbf{B}=0 \\ &\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\ , && \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t} \end{aligned}}$$

Maxwell Equations for Partial Fields

- Now it becomes obvious why the partial fields were defined in different units than the total fields: it makes the constants disappear in the final form of the Maxwell Equations.
- It should be noted that Maxwell's equations give a full description of the electromagnetic *fields.* The complete physical picture is then obtained by using the equation that links the fields to forces (the Lorentz equation) and the one that links forces to accelerations (Newton's law):

$\mathbf{F}=q\,\mathbf{E}+q\,\mathbf{v}\times\mathbf{B}$ or the more general form $\mathbf{F}=\int\left[\rho(\mathbf{x})\mathbf{E}(\mathbf{x})+\mathbf{J}(\mathbf{x})\times\mathbf{B}(\mathbf{x})\right]d^3\mathbf{x}$

$$\mathbf{F}=m\,\mathbf{a}$$

- A quick check of Maxwell's equations reveals that they reduce down to the equations we have been using in electrostatic and magnetostatics if the fields do not depend on time.

3. Boundary Conditions

- The divergence equations are no different than in electrostatics and magnetostatics, so we can apply the Gaussian pillbox method in the usual way and end up with the normal-component boundary conditions:

$$\boxed{(\mathbf{D}_2-\mathbf{D}_1)\cdot\mathbf{n}=\sigma}$$

$$\boxed{(\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n}=0}$$

- Let us now draw an Amperian loop straddling the surface and integrate the curl equations over the loop in the usual way.

$$\oint \mathbf{E}\cdot d\,\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{n}\, da$$

$$\oint \mathbf{H}\cdot d\,\mathbf{l} = \int \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right)\cdot \mathbf{n}\, da$$

- As we shrink the area of the loops to zero, the finite fields go to zero as well. The volume current density **J** becomes a surface current density **K**.

$$\boxed{\mathbf{n}\times(\mathbf{E}_2 - \mathbf{E}_1) = 0}$$

$$\boxed{\mathbf{n}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}}$$

- So it turns out that the boundary conditions for electrodynamics is the same as those we used for electrostatics and magnetostatics.

<u>4. Electromagnetic Waves</u>

- Maxwell's equations form a set of linear, first-order *coupled* differential equations. Using them could be easier if we decouple the equations.
- Take the partial derivative of the Maxwell-Ampere law with respect to time, and put Faraday's law into it:

$$\nabla\times\frac{\partial \mathbf{B}}{\partial t} = \mu_0 \frac{\partial \mathbf{J}_{\text{tot}}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla\times\nabla\times\mathbf{E} = -\mu_0 \frac{\partial \mathbf{J}_{\text{tot}}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial \mathbf{J}_{\text{tot}}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

- Insert in Coulomb's law to find:

$$\boxed{\nabla^2 \mathbf{E} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0}\nabla \rho_{\text{tot}} + \mu_0 \frac{\partial \mathbf{J}_{\text{tot}}}{\partial t}}$$

- A similar process is followed to reveal:

$$\boxed{\nabla^2 \mathbf{B} - \mu_0\epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} = -\mu_0 \nabla\times\mathbf{J}_{\text{tot}}}$$

Maxwell Equations in Wave Form

- The equations above in boxes are Maxwell's equations in wave-equation form. (Coulomb's law and the no-magnetic charge law must still be included with these in order to get a unique

solution.)
- These equations tell us that non-zero fields can exist even in the total absence of charges and currents, in the form of self-propagating electromagnetic waves.
- These equations also tell us that currents and charges, whether bound or free, can create and destroy traveling electromagnetic waves.
- Maxwell's equations in wave-equation form are very useful because all of the field components have been mathematically decoupled (they are of course still coupled physically through ρ and $\mathbf{J}$). This means that each scalar equation contains one and only one field component.
- In view of our experience with waves on a string, we recognize the constants in front of the second-order time derivatives as the inverse of the square of the velocity of the wave. This is found experimentally to be the speed of light in vacuum, c, implying that light is an electromagnetic wave.

$$\frac{1}{c^2} = \mu_0 \epsilon_0$$

- It should be noted that when we left the realm of electrostatics and entered the realm of electrodynamics, we added the extra dimension of time.
- The two known variables, the charge density ρ and the current density $\mathbf{J}$**,** are now functions of four dimensions: (x, y, z, t).
- The two unknown vector fields, the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ are also now functions of the four dimensions.

5. Standard Electrodynamic Vector and Scalar Potential

- Now that we have the most general electrodynamic equations, let us try to use potentials like we used for the special cases of electrostatics and magnetostatics.
- If we approach it properly, the electrodynamics techniques should always reduce to the electrostatics/magnetostatics techniques if the fields are non-varying in time.
- In view of the non-diverging nature of the magnetic field $\nabla \cdot \mathbf{B} = 0$ (which still holds even in electrodynamics) and the fact that the divergence of the curl of some vector is always zero, we can still define the magnetic field in terms of the curl of a vector potential:

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}} \quad \text{because } \nabla \cdot (\nabla \times \mathbf{A}) = 0 \text{ automatically for any } \mathbf{A}$$

- Plugging this into Faraday's Law gives:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \rightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{A}$$

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

- In view of the fact that the curl of a gradient of some scalar is always zero, we can set the terms in parentheses equal to the negative gradient of a scalar potential:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla \Phi \quad \text{because } \nabla \times (\nabla \Phi) = 0 \text{ automatically for any } \Phi$$

$$\boxed{\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}}$$

- If the potential definitions shown in the boxes above are inserted into the remaining Maxwell equations (Coulomb's law and the Maxwell-Ampere law) we find:

$$\nabla^2\Phi+\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}=-\frac{\rho_{\text{total}}}{\epsilon_0}$$

$$\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}+\mu_0\epsilon_0\nabla\frac{\partial\Phi}{\partial t}+\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}=\mu_0\mathbf{J}_{\text{total}}$$

- Due to the nature of the curl and gradient operators, there is some freedom to how we can define the potentials and still end up with the same electromagnetic fields. We must introduce another equation involving the potentials in order to get a unique solution.
- We could pick potentials that obey the Lorenz condition:

$$\nabla\cdot\mathbf{A}=-\mu_0\epsilon_0\frac{\partial\Phi}{\partial t}$$

- Using these particular potentials allows us to decouple the two equations above to find:

$$\boxed{\begin{gathered}\nabla^2\Phi-\mu_0\epsilon_0\frac{\partial^2\Phi}{\partial t^2}=-\frac{\rho_{\text{total}}}{\epsilon_0}\\ \nabla^2\mathbf{A}-\mu_0\epsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}_{\text{total}}\end{gathered}}$$

The Maxwell Equations in Terms of Potentials in the Lorenz Gauge

- This choice of potentials is known as the Lorenz gauge.
- In the Lorenz gauge, the potentials are found to obey wave equations just like the fields.
- Note that the Lorenz gauge is not a completely specified gauge; there is still not a unique solution. More accurately then, the Lorenz condition leads to a family of gauges.
- The equations in boxes above are completely general. If we are in vacuum (there are no dielectric or magnetic materials involved), then of course the total charge density and total current density just reduce to the free charge density and the free current density.
- In uniform, linear, isotropic, dispersionless, lossless dielectric and magnetic materials, this of course reduces to:

$$\nabla^2\Phi-\mu\epsilon\frac{\partial^2\Phi}{\partial t^2}=-\frac{\rho}{\epsilon}$$

$$\nabla^2\mathbf{A}-\mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mathbf{J}$$

- Alternatively, we can choose the gauge where $\nabla \cdot \mathbf{A} = 0$. Then the equations reduce to:

$$\nabla^2 \Phi = -\frac{\rho_{\text{total}}}{\epsilon_0}$$

$$\nabla^2 \mathbf{A} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{J}_{\text{total}} + \mu_0 \epsilon_0 \nabla \frac{\partial \Phi}{\partial t}$$

The Maxwell Equations in Terms of Potentials in the Coulomb Gauge

- This is known as the Coulomb gauge, radiation gauge, or transverse gauge.
- The first equation is the same as in electrostatics. This means that as the charge density changes in time, the scalar electric potential (not the field) responds instantaneously as if it were static.
- This does not violate causality however because only the fields are physical, and the electric fields depends on more than the scalar potential. Beyond the shell of causality, there is always a piece of **A** that will cancel Φ and maintain **E** as non-instantaneously linked to the charge density, thereby preserving causality.