Lecture 12 Notes, Electromagnetic Theory I

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1. Review of Magnetostatics in Magnetic Materials

- Currents give rise to curling magnetic fields:

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}} \quad \text{or} \quad \nabla\times\mathbf{H}=\mathbf{J} \quad \text{or} \quad \nabla\times\mathbf{M}=\mathbf{J}_M \quad \text{where} \quad \mathbf{J}_{\text{total}}=\mathbf{J}+\mathbf{J}_M$$

- There are no magnetic monopoles:

$$\nabla\cdot\mathbf{B}=0 \quad \text{which leads to} \quad \nabla\cdot\mathbf{H}=-\nabla\cdot\mathbf{M}$$

- Defining a vector potential $\mathbf{B}=\nabla\times\mathbf{A}$ leads to:

$$\nabla^2\mathbf{A}=-\mu_0\mathbf{J}_{\text{total}} \quad \text{and} \quad \mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_{\text{total}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

- In a region where the magnetic material is linear and uniform so that $\mathbf{B}=\mu\mathbf{H}$ we can apply all of the **B**-field equations to the free current **J** instead of the total current $\mathbf{J}_{\text{total}}$ if we replace the permittivity of free space μ_0 with the permittivity of the material μ. For instance:

$$\nabla^2\mathbf{A}=-\mu\mathbf{J} \quad \text{and} \quad \mathbf{A}=\frac{\mu}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

- The boundary conditions for any type of materials are:

$$(\mathbf{B}_2-\mathbf{B}_1)\cdot\mathbf{n}=0 \quad \text{and} \quad \mathbf{n}\times(\mathbf{H}_2-\mathbf{H}_1)=\mathbf{K}$$

2. Special Cases in Magnetostatics

- If the materials are linear *and* there is no free current density in the region of space where we want to know the fields (**J** = 0), then the equation reduces to:

$$\boxed{\nabla^2\mathbf{A}=0}$$

- These can be solved in the usual way with appropriate boundary conditions.
- An alternate approach is to define a scalar potential $\mathbf{B}=-\nabla\Psi_M$ so that the zero-divergence equation becomes:

$$\boxed{\nabla^2\Psi_M=0}$$

- If there is no current density, **J** = 0, and if the material is *not* linear, but instead the magnetization **M** is known and fixed (such as in permanent magnets), the equations reduces to:

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}_M} \quad \text{and} \quad \boxed{\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}_M(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\,\mathbf{x}'} \quad \text{where } \mathbf{J}_M = \nabla \times \mathbf{M}$$

- The alternate scalar approach $\mathbf{H} = -\nabla \Phi_M$ can also be used in this case. The statement of no magnetic monopoles really means that the divergence of the **H** field and the **M** field are equal:

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \cdot (\mu_0 \mathbf{H} + \mu_0 \mathbf{M}) = 0$$

$$-\nabla \cdot \mathbf{H} = \nabla \cdot \mathbf{M}$$

$$\boxed{\nabla^2 \Phi_M = \nabla \cdot \mathbf{M}}$$

- where we can now treat the divergence of the magnetization as an effective magnetic charge density: $\rho_M = -\nabla \cdot \mathbf{M}$.

3. Sample Problem

- Consider an unmagnetized solid sphere of radius *a* made of a uniform linear magnetic material with permeability μ which is then placed in an originally uniform external magnetic field $\mathbf{B} = B_0 \hat{\mathbf{z}}$.
- Since there is only linear material and no currents, we can solve for the magnetic potential:

$$\nabla^2 \Psi_M = 0 \quad \text{where } \mathbf{B} = -\nabla \Psi_M \text{ and the potential far away becomes } \Psi_M = -B_0 r \cos\theta$$

- This is simply the Laplace equation in spherical coordinates with azimuthal symmetry, which we already know the general solution to be:

$$\Psi_M(r, \theta, \phi) = \sum_l (A_l r^l + B_l r^{-l-1}) P_l(\cos\theta)$$

- Outside the sphere, apply the boundary condition at large *r*:

$$-B_0 r \cos\theta = \sum_l (A_l r^l) P_l(\cos\theta)$$

$$A_1 = -B_0 \;, \quad A_0 = 0,\; A_2 = 0,\; A_3 = 0 \ldots$$

$$\Psi_M^{\text{out}}(r, \theta, \phi) = -B_0 r \cos\theta + \sum_{l=0}^{\infty} B_l^{\text{out}} r^{-l-1} P_l(\cos\theta)$$

- Inside the sphere, $B_l = 0$ to keep a valid solution at the origin:

$$\Psi_M^{\text{in}}(r, \theta, \phi) = \sum_{l=0}^{\infty} A_l^{\text{in}} r^l P_l(\cos\theta)$$

- There are no currents and all materials are linear, so the boundary conditions become:

$$(\mathbf{B}^{\text{out}}-\mathbf{B}^{\text{in}})\cdot\hat{\mathbf{r}}=0 \quad \text{and} \quad \left(\frac{1}{\mu_0}\mathbf{B}^{\text{out}}-\frac{1}{\mu}\mathbf{B}^{\text{in}}\right)\cdot\hat{\boldsymbol{\theta}}=0 \quad \text{at } r=a$$

- Applying the first boundary condition gives:

$$(-\nabla\Psi_M^{\text{out}}+\nabla\Psi_M^{\text{in}})\cdot\mathbf{r}=0$$

$$\frac{\partial\Psi_M^{\text{out}}}{\partial r}=\frac{\partial\Psi_M^{\text{in}}}{\partial r}$$

$$-B_0\cos\theta+\sum_{l=0}^{\infty}B_l^{\text{out}}(-l-1)a^{-l-2}P_l(\cos\theta)=\sum_{l=1}^{\infty}A_l^{\text{in}}\,l\,a^{l-1}P_l(\cos\theta)$$

$$B_0^{\text{out}}=0$$

$$\boxed{A_1^{\text{in}}=-B_0-2B_1^{\text{out}}a^{-3}}$$

$$\boxed{A_l^{\text{in}}=B_l^{\text{out}}\frac{(-l-1)}{l}a^{-2l-1}}$$

- Applying the second boundary condition gives:

$$\frac{1}{\mu_0}\frac{\partial\Psi_M^{\text{out}}}{\partial\theta}=\frac{1}{\mu}\frac{\partial\Psi_M^{\text{in}}}{\partial\theta}$$

$$\frac{1}{\mu_0}\left[B_0 a\sin\theta+\sum_{l=1}^{\infty}B_l^{\text{out}}a^{-l-1}P'_l(\cos\theta)\right]=\frac{1}{\mu}\left[\sum_{l=1}^{\infty}A_l^{\text{in}}a^{l}P'_l(\cos\theta)\right]$$

$$\boxed{A_1^{\text{in}}=\frac{\mu}{\mu_0}\left[-B_0+B_1^{\text{out}}a^{-3}\right]}$$

$$\boxed{A_l^{\text{in}}=\frac{\mu}{\mu_0}B_l^{\text{out}}a^{-2l-1}}$$

- All of the equations in boxes can only be satisfied if $B_l^{\text{out}}=A_l^{\text{in}}=0$ for $l>1$ and if

$$\boxed{B_1^{\text{out}}=\frac{\mu-\mu_0}{\mu+2\mu_0}a^3B_0}$$

$$\boxed{A_1^{\text{in}}=\left[\frac{-3\mu}{\mu+2\mu_0}\right]B_0}$$

- We now have the final solution to the magnetic potential, and can therefore find the magnetic fields:

$$\Psi_M^{\text{out}}(r,\theta,\phi)=\left[-\left(\frac{r}{a}\right)+\frac{\mu-\mu_0}{\mu+2\mu_0}\left(\frac{a}{r}\right)^2\right]B_0 a\cos\theta \quad \text{and} \quad \Psi_M^{\text{in}}(r,\theta,\phi)=\left[\frac{-3\mu}{\mu+2\mu_0}\right]B_0 r\cos\theta$$

$$\mathbf{B}=-\nabla\Psi_M$$

$$\mathbf{B}=-\left[\hat{\mathbf{r}}\frac{\partial\Psi_M}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\Psi_M}{\partial\theta}\right]$$

$$\mathbf{B}^{\text{in}}=\left[\frac{3\mu}{\mu+2\mu_0}\right]B_0\left[\hat{\mathbf{r}}\cos\theta-\hat{\boldsymbol{\theta}}\sin\theta\right]$$

$$\boxed{\mathbf{B}^{\text{in}}=\left[\frac{3\mu}{\mu+2\mu_0}\right]B_0\hat{\mathbf{z}}} \quad \text{and} \quad \boxed{\mathbf{B}^{\text{out}}=B_0\hat{\mathbf{z}}+B_0\frac{\mu-\mu_0}{\mu+2\mu_0}\left(\frac{a}{r}\right)^3\left[3\hat{\mathbf{r}}\cos\theta-\hat{\mathbf{z}}\right]}$$

and the other fields are linearly related:

$$\mathbf{H}^{\text{in}}=\frac{1}{\mu}\mathbf{B}^{\text{in}} \quad\rightarrow\quad \boxed{\mathbf{H}^{\text{in}}=\left[\frac{3}{\mu+2\mu_0}\right]B_0\hat{\mathbf{z}}}$$

$$\mathbf{H}^{\text{out}}=\frac{1}{\mu_0}\mathbf{B}^{\text{out}} \quad\rightarrow\quad \boxed{\mathbf{H}^{\text{out}}=\frac{1}{\mu_0}B_0\hat{\mathbf{z}}+B_0\frac{\mu-\mu_0}{\mu_0(\mu+2\mu_0)}\left(\frac{a}{r}\right)^3\left[3\hat{\mathbf{r}}\cos\theta-\hat{\mathbf{z}}\right]}$$

$$\mathbf{M}^{\text{in}}=\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)\mathbf{B}^{\text{in}} \quad\rightarrow\quad \boxed{\mathbf{M}^{\text{in}}=\left[\frac{3(\mu-\mu_0)}{\mu_0(\mu+2\mu_0)}\right]B_0\hat{\mathbf{z}}}$$

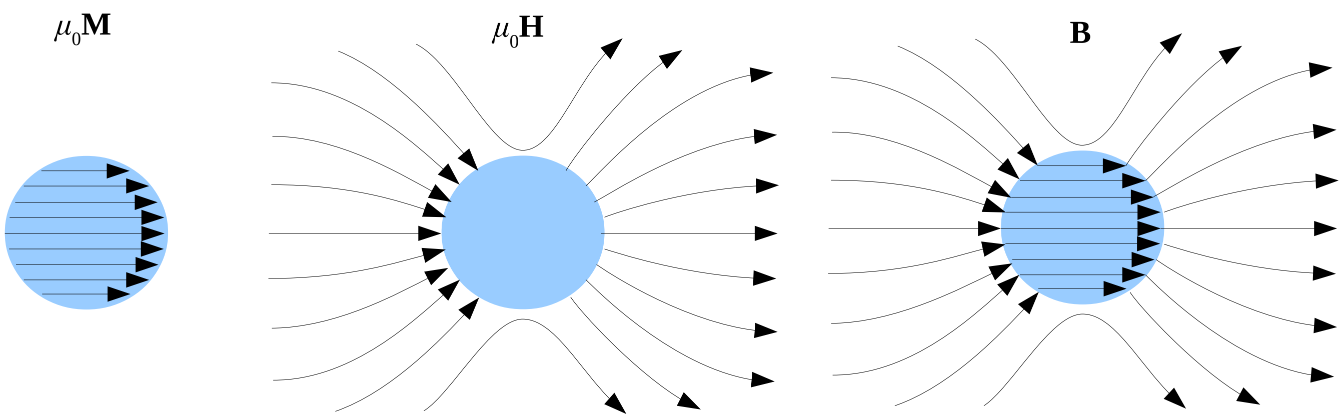
$$\mathbf{M}^{\text{out}}=\left(\frac{1}{\mu_0}-\frac{1}{\mu_0}\right)\mathbf{B}^{\text{out}} \quad\rightarrow\quad \boxed{\mathbf{M}^{\text{out}}=0}$$

- We can check this by looking at several special cases.
- If the permeability of the sphere is just the permeability of free space, $\mu=\mu_0$, the sphere is essentially removed and there should be no fields except the original uniform field. The equations above confirm this.

- If the permeability of the sphere is positively infinite (a perfectly paramagnetic material), the fields reduce to:

$$\mathbf{H}^{\text{in}}=0 \quad , \quad \mathbf{B}^{\text{in}}=\mu_0\mathbf{M}^{\text{in}}=3B_0\hat{\mathbf{z}}$$

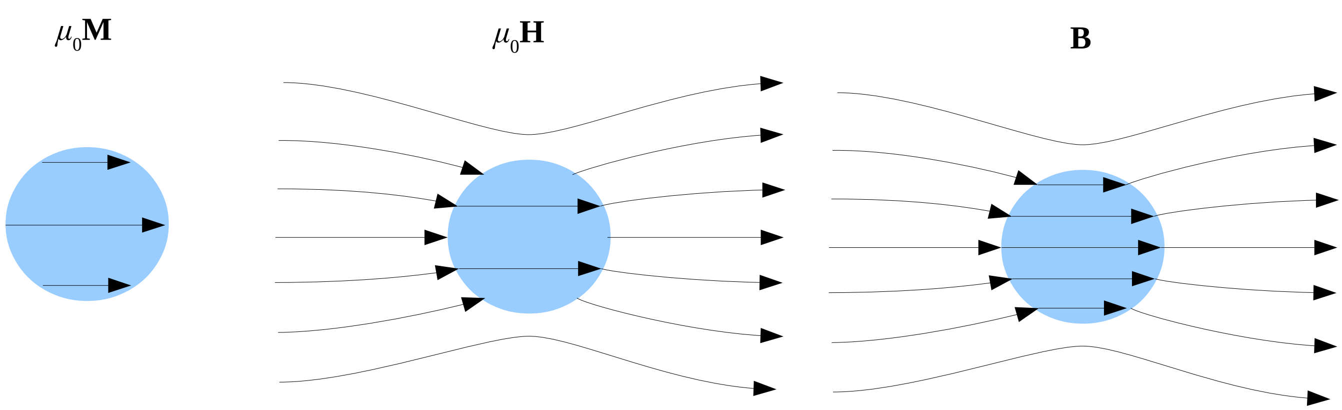
$$\mathbf{M}^{\text{out}}=0 \quad , \quad \mathbf{B}^{\text{out}}=\mu_0\mathbf{H}^{\text{out}}=B_0\hat{\mathbf{z}}+B_0\left(\frac{a}{r}\right)^3[3\hat{\mathbf{r}}\cos\theta-\hat{\mathbf{z}}] \quad \text{(at } r=a\text{, we have } \mathbf{B}^{\text{out}}=3B_0\cos\theta\,\hat{\mathbf{r}}\text{)}$$



- If the permeability is twice the permeability of free space (weak paramagnetic material), $\mu = 2\mu_0$ then we have:

$$\mu_0\mathbf{M}^{\text{in}}=\frac{3}{4}B_0\hat{\mathbf{z}} \quad , \quad \mu_0\mathbf{H}^{\text{in}}=\frac{3}{4}B_0\hat{\mathbf{z}} \quad , \quad \mathbf{B}^{\text{in}}=\frac{3}{2}B_0\hat{\mathbf{z}}$$

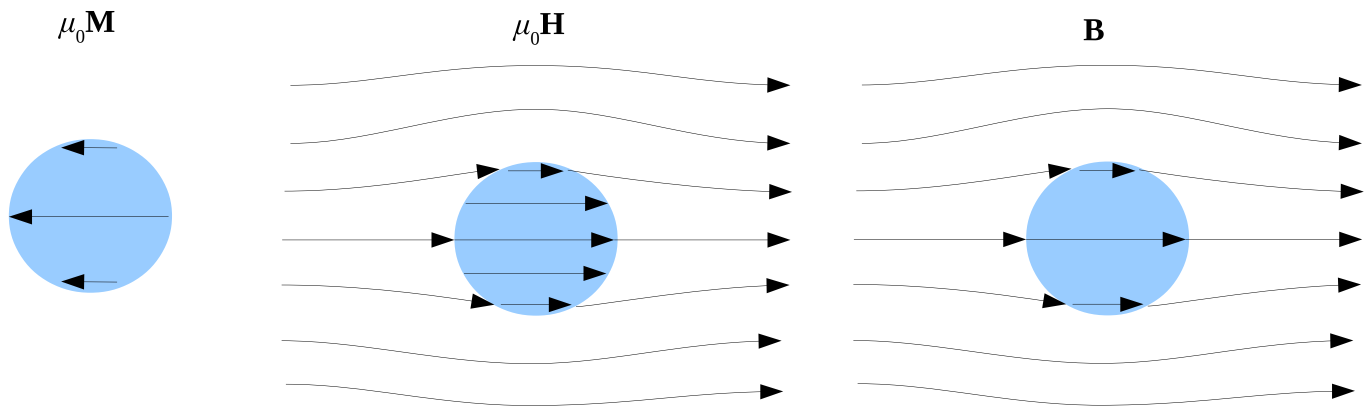
$$\mathbf{M}^{\text{out}}=0 \quad , \quad \mathbf{B}^{\text{out}}=\mu_0\mathbf{H}^{\text{out}}=B_0\hat{\mathbf{z}}+B_0\frac{1}{4}\left(\frac{a}{r}\right)^3[3\hat{\mathbf{r}}\cos\theta-\hat{\mathbf{z}}]$$



- If the permeability is half the permeability of free space (diamagnetic material), $\mu = (1/2)\mu_0$ then we have:

$$\mu_0 \mathbf{M}^{\text{in}} = -\frac{3}{5} B_0 \hat{\mathbf{z}} \quad , \quad \mu_0 \mathbf{H}^{\text{in}} = \frac{6}{5} B_0 \hat{\mathbf{z}} \quad , \quad \mathbf{B}^{\text{in}} = \frac{3}{5} B_0 \hat{\mathbf{z}}$$

$$\mathbf{M}^{\text{out}} = 0 \qquad \mathbf{B}^{\text{out}} = \mu_0 \mathbf{H}^{\text{out}} = B_0 \hat{\mathbf{z}} - B_0 \frac{1}{5} \left(\frac{a}{r}\right)^3 [3 \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}}]$$



4. Faraday's Law of Induction

- We now leave magnetostatics and ask the question: What is the effect if the magnetic field is changing in time?
- Faraday observed that moving magnets, moving wires with currents, or changing the current in wires created currents in near-by wires.
- Physically this means that magnetic fields that change in time create electric fields, which exert a force and drive electric currents through adjacent wires.
- The change in time of the total magnetic flux F_B through a surface S creates an electromotive force E around the loop C bounding S according to:

$$E = -\frac{dF_B}{dt}$$

- The negative sign accounts for the fact that the changing magnetic flux induces currents that create fields which oppose the original flux.
- Expanding the flux and electromotive force in terms of their integral definitions:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{n} \, da$$

- The time derivative can be expanded into partial derivatives. The additional terms can be combined with the electric field if we redefine the electric field as fixed in the laboratory frame and not fixed on the moving loop C.

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{n}\, da$$

Faraday's Law in Integral Form

- Apply Stoke's Theorem to the left side:

$$\int_S (\nabla \times \mathbf{E})\cdot \mathbf{n}\, da = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot \mathbf{n}\, da$$

- Shrink the surface S until it is infinitesimally small so that the integrands must hold at all points in space:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0$$

Faraday's Law in Differential Form

- Note that if the magnetic field **B** is constant in time, this reduces to $\nabla \times \mathbf{E} = 0$ which was the case in electrostatics. Physically this means that changing magnetic fields can create curling electric fields.

5. Energy in the Magnetic Field

- An electromotive force E does work W on a current I inside a loop according to:

$$\frac{dW}{dt} = -I\, E$$

- A changing magnetic flux F_B creates an electromotive force and thus does work:

$$\frac{dW}{dt} = I \frac{dF_B}{dt}$$

- For an instantaneous time interval:

$$dW = I\, dF_B$$

- This can be cast in terms of the magnetic fields and after much algebra we find for linear materials:

$$W = \frac{1}{2}\int \mathbf{H}\cdot \mathbf{B}\, d^3\mathbf{x}$$

$$W = \frac{1}{2\mu}\int |\mathbf{B}|^2\, d^3\mathbf{x}$$

- This means that the total magnetic potential energy stored in a system is equal to the integral of the magnetic field squared.

- We can then speak of a magnetic potential energy density w: $w = \frac{1}{2\mu}|\mathbf{B}|^2$