C. S. BAIRD

Lecture 11 Notes, Electromagnetic Theory I

Dr. Christopher S. Baird
University of Massachusetts Lowell

UMASS LOWELL

1. Magnetic Fields of a Localized Current Distribution

- Similar to the electrostatic multipole expansion, if we have a localized current distribution and want to know the magnetic fields far away, we can make a magnetostatic multiplole expansion and only keep the first few terms.
- Consider the Biot-Savart Law for the vector potential:

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\,\mathbf{x}' \quad \text{where } \mathbf{B}(\mathbf{x})=\nabla\times\mathbf{A} \text{ and this is a solution to } \nabla\times\mathbf{B}(\mathbf{x})=\mu_0\,\mathbf{J}(\mathbf{x})$$

- The primed coordinates refer to the localized currents, so that an expansion in powers of the primed variables will allow us to drop higher order terms:

$$\frac{1}{|\mathbf{x}-\mathbf{x}'|}=\frac{1}{|\mathbf{x}|}+\frac{\mathbf{x}\cdot\mathbf{x}'}{|\mathbf{x}|^3}+\ldots$$

- Using this expansion:

$$\mathbf{A}=\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|}\int\mathbf{J}(\mathbf{x}')\,d\,\mathbf{x}'+\frac{\mu_0}{4\pi}\frac{\mathbf{x}}{|\mathbf{x}|^3}\cdot\int\mathbf{x}'\,\mathbf{J}(\mathbf{x}')\,d\,\mathbf{x}'+\ldots$$

- The first term corresponds to the magnetic field generated by a magnetic monopole moment. We already know that there are no monopoles, so that this term should be zero. Let us prove it.
- Consider a functions f and g and make an expansion using integration by parts:

$$\int_V g\,\mathbf{J}\cdot\nabla' f\,d\,\mathbf{x}'=\int_S f\,g\,\mathbf{J}\cdot\hat{\mathbf{x}}'\,da-\int_V f\,\nabla'\cdot(g\,\mathbf{J})\,d\,\mathbf{x}'$$

- Because the current is localized, if the bounding surface is larger than the current distribution, the current density will be everywhere zero on the surface, so that:

$$\int_V g\,\mathbf{J}\cdot\nabla' f\,d\,\mathbf{x}'=-\int_V f\,\nabla'\cdot(g\,\mathbf{J})\,d\,\mathbf{x}'$$

- Expand the right side using a vector identity:

$$\int_V g\,\mathbf{J}\cdot\nabla' f\,d\,\mathbf{x}'=-\int_V f\,\mathbf{J}\cdot\nabla' g\,d\,\mathbf{x}'-\int_V f\,g\,\nabla'\cdot\mathbf{J}\,d\,\mathbf{x}'$$

- Collect terms:

$$\int_V(g\,\mathbf{J}\cdot\nabla' f+f\,\mathbf{J}\cdot\nabla' g+f\,g\,\nabla'\cdot\mathbf{J})\,d\,\mathbf{x}'=0$$

- This holds for any functions f and g and vector **J**. We now apply it to our problem. Make **J** the current density vector. The definition of magnetostatics is that there is no divergence: $\nabla' \cdot \mathbf{J} = 0$

$$\int_V (g\,\mathbf{J}\cdot\nabla' f + f\,\mathbf{J}\cdot\nabla' g)\, d\mathbf{x}' = 0$$

- Set $f = 1$ and $g = x_i$:

$$\int_V J_i\, d\mathbf{x}' = 0$$

- If the integral of every component is zero, than the integral of the whole vector is zero and we have proven that the first term in the expansion is zero. If only the second term in the expansion is kept and all higher order terms are dropped, the potential far from a localized current density becomes:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int (\mathbf{x}\cdot\mathbf{x}')(\mathbf{J}(\mathbf{x}'))\, d\mathbf{x}'$$

- Express this explicitly in terms of components:

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int \left(\sum_j x_j x_j'\right)\left(\sum_i \hat{\mathbf{x}}_i J_i\right) d\mathbf{x}'$$

- Break the terms in half and shuffle things around:

$$\mathbf{A} = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int d\mathbf{x}' \sum_i \hat{\mathbf{x}}_i \sum_j x_j [(-x_j' J_i) - x_j' J_i]$$

- If we set $f = x_i'$ and $g = x_j'$ in the identity on the bottom of the previous page, we get:

$$\int_V (-x_j' J_i)\, d\mathbf{x}' = \int_V (x_i' J_j)\, d\mathbf{x}'$$

- Use this to evaluate the integral in the potential equation:

$$\mathbf{A} = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int d\mathbf{x}' \sum_i \hat{\mathbf{x}}_i \sum_j x_j [x_i' J_j - x_j' J_i]$$

$$\begin{aligned}\mathbf{A} = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int d\mathbf{x}' (&\hat{\mathbf{x}}[y(x' J_y - y' J_x) - z(z' J_x - x' J_z)] \\ &+ \hat{\mathbf{y}}[z(y' J_z - z' J_y) - x(x' J_y - y' J_x)] \\ &+ \hat{\mathbf{z}}[x(z' J_x - x' J_z) - y(y' J_z - z' J_y)])\end{aligned}$$

$$\mathbf{A} = -\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int d\mathbf{x}'\, \mathbf{x}\times((y' J_z - z' J_y)\hat{\mathbf{x}} + (z' J_x - x' J_z)\hat{\mathbf{y}} + (x' J_y - y' J_x)\hat{\mathbf{z}})$$

$$\mathbf{A}=-\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\int d\,\mathbf{x}'\,\mathbf{x}\times(\mathbf{x}'\times\mathbf{J})$$

$$\boxed{\mathbf{A}=-\frac{1}{2}\frac{\mu_0}{4\pi}\frac{1}{|\mathbf{x}|^3}\mathbf{x}\times\int(\mathbf{x}'\times\mathbf{J})\,d\,\mathbf{x}'}$$

Potential far away from localized current distribution

- This looks very similar in form to the potential produced by an electric dipole if we define a magnetic dipole moment **m**:

$$\boxed{\mathbf{A}=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}} \quad \text{where} \quad \boxed{\mathbf{m}=\frac{1}{2}\int(\mathbf{x}'\times\mathbf{J})\,d\,\mathbf{x}'}$$

Potential of a magnetic dipole

- The magnetic field is easily calculated:

$$\mathbf{B}(\mathbf{x})=\nabla\times\mathbf{A}$$

$$\mathbf{B}(\mathbf{x})=\frac{\mu_0}{4\pi}\nabla\times\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3}$$

$$\boxed{\mathbf{B}(\mathbf{x})=\frac{\mu_0}{4\pi}\frac{3\,\mathbf{n}(\mathbf{n}\cdot\mathbf{m})-\mathbf{m}}{|\mathbf{x}|^3}}$$

Magnetic field of a magnetic dipole

2. Behavior of a Localized Current Density in the Presence of an External Magnetic Field

- If an external magnetic field is applied to a localized current distribution, the distribution feels a force and a torque.
- Assume the external magnetic field is slowly varying over the distribution. We can make a Taylor series expansion of the magnetic field and only keep the lowest terms.
- After some algebra, we find:

$$\boxed{\mathbf{F}=\nabla(\mathbf{m}\cdot\mathbf{B}(0))}$$

- A constant magnetic field exerts no net force on a localized current distribution.
- Non-uniform fields *do* exert a force on localized current distributions.
- In view of $\mathbf{F}=q\,\mathbf{v}\times\mathbf{B}$, free charged particles tend to spiral around magnetic field lines using the right hand rule. The spiraling motion creates a current loop with the magnetic moment generally pointing in the same direction as the magnetic field. If the spiraling particle approaches a region where the magnetic fields line pinch together so that the field increases, the gradient and thus the force become non-zero and point away from high-flux region. This is the concept behind magnetic confinement and the reason that the Van Allen radiation belts have low density over the earth's magnetic poles.
- The torque ends up as:

$$\boxed{\mathbf{N}=\mathbf{m}\times\mathbf{B}(0)}$$

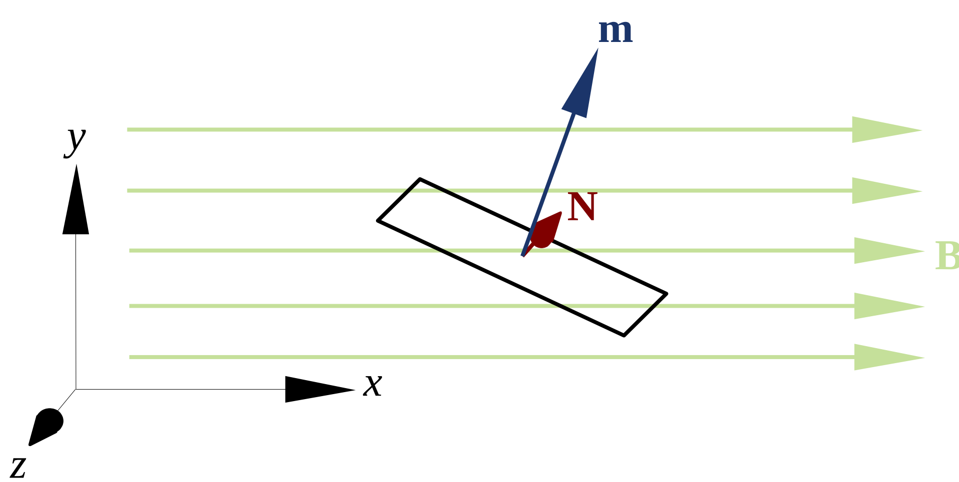
- Suppose the external magnetic field is uniform and we arrange the axes so that the magnetic field points in the x direction, $\mathbf{B}(0) = B_0 \hat{\mathbf{i}}$, and the magnetic moment is in the x-y plane,

$$\mathbf{m} = m(\cos\theta_m \hat{\mathbf{i}} + \sin\theta_m \hat{\mathbf{j}})$$

$$\mathbf{F} = 0$$

$$\mathbf{N} = m(\cos\theta_m \hat{\mathbf{i}} + \sin\theta_m \hat{\mathbf{j}}) \times (B_0 \hat{\mathbf{i}})$$

$$\mathbf{N} = -B_0 m \sin\theta_m \hat{\mathbf{k}}$$



3. Magnetostatics in Ponderable Materials

- Up to this point, all the magnetostatics have been done in a vacuum.
- If the currents are placed in a material, the material responds and adds addition magnetic effects.
- We must than add these additional magnetic effects to the equations above.
- Analogous to the electric polarization of a material, the magnetization **M** is the magnetic moment density.
- The magnetization may be induced or may exist of itself such as in a permanent magnet.
- Macroscopically speaking, a magnetic material contains many small regions of non-zero magnetic moment called domains. (The magnetic moments in a domain are ultimately caused by the orbits of electrons in molecules causing tiny loop currents, but a quantum approach is required to describe this realm.)
- In a non-permanent magnet, the magnetic moments of the domains point randomly and cancel each other out on average.
- When an external magnetic field is applied or a current density is embedded in the material, creating an applied magnetic field, the domains feel a torque until they are aligned. They no longer cancel out but create an additional magnetic field known as the magnetization **M**.
- There has traditionally been a wide range of names applied to the different magnetic fields which can lead to confusion. It is safest to simply understand what each field is physically and than refer to each by its letter name.

- For clarity, let us define the following:

$\mu_0\mathbf{H}$: The *applied* magnetic field due to the free currents, plus the magnetic field due to effective magnetic charges (where **H** is known historically as the "magnetizing field", "auxiliary magnetic field", "magnetic field intensity", or even just "magnetic field").
$\mu_0\mathbf{M}$: The *induced* or *permanent* magnetic field caused by the magnetization of the material, or in other words due to the bound currents (where **M** is the magnetization)
B: The *total* magnetic field including the applied magnetic field and the material's response to the applied field/innate magnetization (**B** is also known historically as the "magnetic flux density" or "magnetic induction".)

- Instead of treating all of the fields as the result of electric currents external to the problem, we can explicitly include them. We can define the following:

J: The *free current* density, which gives rise to the curling of the applied magnetic field $\mu_0\mathbf{H}$ (a non-curling **H** field can exist even when there are no free currents).
$\mathbf{J}_M$: The *magnetization current* density, (induced or permanent bound currents), which gives rise to the magnetization **M**.
$\mathbf{J}_{\text{total}}$: The *total current* density, which is a combination of the free currents and magnetization, and gives rise to the total magnetic field **B**.

- As derived previously, the magnetic vector potential due to a localized current distribution can be approximated by the first non-zero term in the expansion, the dipole term:

$$\mathbf{A}=\frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{x}}{|\mathbf{x}|^3} \quad \text{(for a single dipole)}$$

- To include the effects of materials, we simply integrate over all of the dipoles' potentials and add this result to the potential due to the free currents found above:

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\,\mathbf{x}'+\frac{\mu_0}{4\pi}\int\frac{\mathbf{M}\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}d^3\mathbf{x}'$$

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\left[\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}+\frac{\mathbf{M}\times(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\right]d\,\mathbf{x}'$$

- After an integration by parts and the use of a vector identity, this can be cast in the form:

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')+\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\,\mathbf{x}'$$

- This can be treated as the original expression if we define the following:

$$\boxed{\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}_{\text{total}}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\,\mathbf{x}'} \quad \text{where} \quad \boxed{\mathbf{J}_{\text{total}}=\mathbf{J}+\mathbf{J}_M} \quad \text{and} \quad \boxed{\mathbf{J}_M=\nabla\times\mathbf{M}}$$

- We can now use all the equations from the magnetostatics-in-vacuum analysis for the magnetostatics-in-material case if we recognize the currents in each equation as now the total current density.
- Ampere's Law in differential form therefore becomes:

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}_{\text{total}}$$

$$\boxed{\nabla\times\mathbf{B}=\mu_0[\mathbf{J}+\nabla\times\mathbf{M}]}$$

- Just as the magnetization current density $\mathbf{J}_M$ gives rise to the magnetization $\mathbf{J}_M=\nabla\times\mathbf{M}$, the free current density gives rise to the applied magnetic field according to $\mathbf{J}=\nabla\times\mathbf{H}$. Plugging this
into Ampere's law gives us:

$$\nabla\times\mathbf{B}=\mu_0[\nabla\times\mathbf{H}+\nabla\times\mathbf{M}]$$

$$\boxed{\mathbf{B}=\mu_0\mathbf{H}+\mu_0\mathbf{M}}$$

- We have thus shown mathematically that the total field equals the effects of free currents and material magnetic effects.
- The analysis thus far is completely general and applies to all magnetic materials.
- The equations simplify quite a bit for certain special types of magnetic materials.

4. Magnetostatics in Isotropic, Linear, Uniform Diamagneic/Paramagnetic Materials

-Certain materials have a linear connection between the applied field and the induced field:

$\mathbf{M}=(\mu_r-1)\mathbf{H}$ where μ_r is the relative magnetic permeability, $\mu_r=\mu/\mu_0$

- In free space, $\mu = \mu_0$, $\mu_r = 1$, and so $\mathbf{M} = 0$. This means $\mathbf{B}=\mu_0\mathbf{H}$ in free space regions.
- In paramagnetic materials, $\mu > \mu_0$, $\mu_r > 1$, and so $\mathbf{M} = + |\mu_r - 1|\ \mathbf{H}$. The magnetization points in the same direction and amplifies the applied field. Paramagnetic materials such as steel or iron are always attracted to high-flux regions such as the tip of a permanent magnet.
- In diamagnetic materials, $\mu < \mu_0$, $\mu_r < 1$, and so $\mathbf{M} = - |\mu_r - 1|\ \mathbf{H}$. The magnetization points in the opposite direction as the applied field and weakens the applied field. Diamagnetic materials are repelled away from regions of high flux such as the tip of a permanent magnet.
- The higher the absolute value of the magnetic permeability, the more a material responds to an applied field.
- Plug the linear relationship above into all of the material magnetostatic equations and they simplify:

$$\mathbf{B}=\mu_0\mathbf{H}+\mu_0\mathbf{M} \quad\rightarrow\quad \boxed{\mathbf{B}=\mu\mathbf{H}}$$

$$\mathbf{A}=\frac{\mu_0}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')+\nabla'\times\mathbf{M}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}' \quad\rightarrow\quad \boxed{\mathbf{A}=\frac{\mu}{4\pi}\int\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'}$$

$$\nabla\times\mathbf{B}=\mu_0[\mathbf{J}+\nabla\times\mathbf{M}] \quad\rightarrow\quad \boxed{\nabla\times\mathbf{B}=\mu\mathbf{J}}$$

$$\nabla^2 \mathbf{A} = -\mu_0 [\mathbf{J} + \nabla \times \mathbf{M}] \quad \rightarrow \quad \boxed{\nabla^2 \mathbf{A} = -\mu \mathbf{J}(\mathbf{x})}$$

- Therefore, the total magnetic field in the presence of linear magnetic materials can be found as if the magnetic material is not even there, as long as the permeability of free space is replaced with the permeability of the material, but only in a region of uniform material.

5. Boundary Conditions on the Magnetic Fields

- We can use the uniform material approach even if there are multiple materials present.
- We can divide the problem into different regions that meet at boundaries, and than we can break the problem into many smaller problems. We solve the equations in each region, than match up the solutions at the boundary to get the final, unique solution.
- We wish then to derive the boundary conditions on the magnetic fields at a boundary where two different materials meet.
- Start with the statement that there are no magnetic monopoles:

$$\nabla \cdot \mathbf{B} = 0$$

- Integrate this over a pillbox volume V which is half in one material and half in the other. Use the divergence theorem to convert the volume integral to a surface integral:

$$\oint_S \mathbf{B} \cdot \mathbf{n}\, da = 0$$

- Shrink the volume V until its sides make no contribution to the integral and the magnetic field is constant over the surface:

$$\boxed{(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0}$$

- Conceptually this means that the normal component of the total magnetic field is continuous across material boundaries. (The normal points from region 1 into region 2.)
- Now start with the statement that the free currents give rise to the **H** field:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

- Integrate this over a square surface S half in one material and half in the other, then use Stoke's theorem to convert the surface integral of a curl to a line integral along the boundary of the S.

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot \mathbf{t}\, da$$

- The normal vector **t** is normal to the surface S and is therefor tangent to the material boundary.
- Shrink the rectangular loop until the sides give no contribution to the integral and the **H** field and current are constant over the top and bottom. The loop vector **l**, the boundary surface normal **n** and the boundary surface tangent **t** are all perpendicular so that $\mathbf{l} = \mathbf{t} \times \mathbf{n}$:

$$\mathbf{H}_2 \cdot (\mathbf{t} \times \mathbf{n}) L_1 - \mathbf{H}_1 \cdot (\mathbf{t} \times \mathbf{n}) L_1 = \mathbf{J} \cdot \mathbf{t}\, L_1 L_2$$

- The current density contained in the two-dimensional boundary surface times a unit length is known as the surface current density $\mathbf{K} = \mathbf{J} L_2$ so that we now have:

$$(\mathbf{H}_2 - \mathbf{H}_1)\cdot(\mathbf{t}\times\mathbf{n}) = \mathbf{K}\cdot\mathbf{t}$$

- Use of a vector identity gives:

$$(\mathbf{n}\times(\mathbf{H}_2 - \mathbf{H}_1))\cdot\mathbf{t} = \mathbf{K}\cdot\mathbf{t}$$

- The vector **t** normal to the integration surface is arbitrary and can be oriented in any direction within the surface plane. This equation must therefore hold for all possible directions:

$$\boxed{\mathbf{n}\times(\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{K}}$$

- Conceptually this means that the free surface current gives rise to a jump in the tangential component of the **H** field across the material's boundary.