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Lecture 10 Notes, Electromagnetic Theory I

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1. Magnetostatics Introduction

- All of the proceeding concepts have been applied to electrostatics: when static electric charges create static electric fields.
- We now turn to a special case of electrodynamics known as magnetostatics: when electric charges move, but move in such a way that they create static magnetic fields.
- The current density \mathbf{J} is a vector field that describes the flow of charge at every point in space.
- The current density is measured as the amount of positive charge per unit area per unit time, with the direction of the vector indicating the direction of charge flow.
- When the electric current is confined in a wire, it is useful to integrate the current density over the cross-sectional area of the wire and find the total current I , or total electric charge flowing through the wire per unit time.
- If the total charge Q_V inside a volume V decreases, this means that some charge is flowing out passed the surface S bounding the volume. Charge is always conserved, it can be neither created nor destroyed:

$$-\frac{\partial Q_V}{\partial t} = \oint_S \mathbf{J} \cdot \mathbf{n} da$$

- Use of the divergence theorem leads to the form:

$$-\frac{\partial Q_V}{\partial t} = \int_V \nabla \cdot \mathbf{J} d\mathbf{x}$$

- Expand the total charge:

$$-\frac{\partial}{\partial t} \int_V \rho d\mathbf{x} = \int_V \nabla \cdot \mathbf{J} d\mathbf{x}$$

- Shrink the volume down so that the integrands must be equal everywhere:

$$\boxed{-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J}}$$

- This is the continuity equation in differential form.
- Magnetostatics is the special case where we assume no build up or depletion of charge at any point:

$$\boxed{\nabla \cdot \mathbf{J} = 0}$$

Definition of Magnetostatics (constant flow of charge)

- The magnetic induction field \mathbf{B} (also known as the magnetic flux density) is a vector field created by electrical currents. Magnetic fields directly produce forces on magnets.

2. The Biot-Savart Law

- Using *small* straight wires containing currents and compass magnets, Oersted, Biot, and Savart experimentally found the following properties:
- The magnetic field is directly proportional to the length dl of the small wire.
- The magnetic field is directly proportional to the electrical current I in the wire.
- The magnetic field is inversely proportional to the square of the distance r from the wire.
- The magnetic field points in the direction normal to the plane in which the wire and observation point lie.
- Expressing each of these experimental observations one by one:

$$dB \propto dl$$

$$dB \propto I dl$$

$$dB \propto \frac{I dl}{r^2}$$

$$d\mathbf{B} \propto \frac{I dl}{r^2} (\hat{\mathbf{l}} \times \hat{\mathbf{r}})$$

$$d\mathbf{B} = k \frac{I dl}{r^2} (\hat{\mathbf{l}} \times \hat{\mathbf{r}})$$

- Combine the vector magnitudes with the unit vectors to get total vectors:

$$d\mathbf{B} = k \frac{I}{r^3} (d\mathbf{l} \times \mathbf{r})$$

- The proportionality constant k depends on the units used. In SI units, $k = \mu_0 / 4\pi$ where μ_0 is the magnetic permeability of free space, so that the equation becomes:

$$d\mathbf{B} = \frac{\mu_0}{4\pi} \frac{I}{r^3} (d\mathbf{l} \times \mathbf{r})$$

- The total magnetic field generated by an electric current flowing through a wire is the integral of the field generated by all of its small segments:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l} \times \mathbf{r}}{r^3}$$

The Biot-Savart Law for Wires

- But what about more complex electric currents that are not confined to small wires?
- We expand the current I in terms of its integration over the current density \mathbf{J} :

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \int \frac{\mathbf{J} \cdot \hat{\mathbf{l}} d\mathbf{l} \times \mathbf{r}}{r^3} da$$

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int \int \frac{\mathbf{J} \times \mathbf{r}}{r^3} dl da$$

- Combine the integral along the wire and the integral over the cross-section of the wire to get a total volume integral. Also expand the separation vector \mathbf{r} in terms of fixed coordinates, $\mathbf{r} = \mathbf{x} - \mathbf{x}'$:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}') \times (\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} d^3 \mathbf{x}'$$

The Biot-Savart Law

- The Biot-Savart Law is the magnetic equivalent of Coulomb's law. Just like Coulomb's law, the integral in this equation must be done over all space. This renders it not particularly useful. We seek to convert it to a form that is local and can accommodate bounding surfaces.

3. Ampere's Law

- As with the the electrostatics case, let us try to get this magnetostatics equation in differential form.

- Use the identity $\frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} = -\nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right)$:

$$\mathbf{B}(\mathbf{x}) = \frac{-\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \times \left(\nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \right) d^3 \mathbf{x}'$$

- Use the identity $-\mathbf{J} \times (\nabla \psi) = \nabla \times (\psi \mathbf{J}) - \psi (\nabla \times \mathbf{J})$:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \left[\nabla \times \left(\left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) \mathbf{J}(\mathbf{x}') \right) - \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) (\nabla \times \mathbf{J}(\mathbf{x}')) \right] d^3 \mathbf{x}'$$

- In this case, $\nabla \times \mathbf{J}(\mathbf{x}') = 0$ for the simple reason that the current density \mathbf{J} is defined entirely in terms of the primed dimensions here and the curl contains derivatives with respect to the unprimed dimensions.

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \int \nabla \times \left(\frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \right) d^3 \mathbf{x}'$$

- The curl is in the unprimed system and the integral is over the independent primed system, so that the curl operation can be moved out of the integral.

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3 \mathbf{x}'$$

- Taking the divergence of both side and noting that the divergence of the curl is always zero leads to the observation that $\nabla \cdot \mathbf{B} = 0$. Remembering that the divergence of the electric field is caused by electric charge, we conclude that a non-zero divergence in the magnetic field would be caused by magnetic charge.

- The fact that the magnetic field never diverges means that *there are no magnetic charges*.
- This means that ultimately all magnetic fields are created by moving electric charges, even if only on the atomic level.
- This means also that magnetic field **B** lines have no beginning or end but must always join into loops.
- If we instead take the curl of both sides, we get:

$$\nabla \times \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

- Use the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

$$\nabla \times \mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \int \mathbf{J}(\mathbf{x}') \cdot \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' - \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{x}') \nabla^2 \frac{1}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

- Use the identities: $\nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -\nabla' \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right)$, $\nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} \right) = -4\pi \delta(\mathbf{x} - \mathbf{x}')$

$$\nabla \times \mathbf{B}(\mathbf{x}) = \frac{-\mu_0}{4\pi} \nabla \int \mathbf{J}(\mathbf{x}') \cdot \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' + \mu_0 \int \mathbf{J}(\mathbf{x}') \delta(\mathbf{x} - \mathbf{x}') d\mathbf{x}'$$

$$\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x}) - \frac{\mu_0}{4\pi} \nabla \int \mathbf{J}(\mathbf{x}') \cdot \nabla' \frac{1}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

- Integration by parts gives:

$$\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x}) - \frac{\mu_0}{4\pi} \nabla \int \nabla' \cdot \mathbf{J}(\mathbf{x}') \frac{1}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

- In magnetostatics, by definition there is no charge build up so that $\nabla \cdot \mathbf{J} = 0$ giving:

$$\boxed{\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x})}$$

Ampere's Law in Differential Form

- This equation means that electric currents give rise to curling magnetic fields. If there is no current present, there can still be a non-zero magnetic field. It is simply irrotational and determined completely by the boundary conditions.
- In electrostatics, we could draw a Gaussian surface and convert Gauss's law back and forth between differential and integral form. In magnetostatics, we can also convert Ampere's law back and forth between differential and integral form.
- Consider an arbitrarily curved surface S bounded by the line contour C . The magnetic field **B** is defined everywhere on S in terms of the current density **J** according to Ampere's Law in differential form.
- Dot both sides with the unit normal vector on the surface S at point \mathbf{x} and integrate over all such points on the surface.

$$\int_S (\nabla \times \mathbf{B}) \cdot \mathbf{n} da = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} da$$

- Apply Stoke's Theorem to get:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int_S \mathbf{J} \cdot \mathbf{n} da$$

- The surface integral of the electric current density is just the total current passing through the loop:

$$\oint_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}$$

Ampere's Law in Integral Form

- The closed contour C is called an Amperian loop and is similar to the Gaussian surfaces of electrostatics.

4. Magnetic Force

- In the physical world, we really only measure forces.
- In statics, electric and magnetic fields are ways of generalizing the effect created by charges and currents to all points of space. (In electrodynamics, fields take on an existence of their own independent of sources.)
- Experimentally, it is found that the magnetic field \mathbf{B} exerts a force \mathbf{F} on a test charge q with velocity \mathbf{v} according to:

$$\mathbf{F} = q \mathbf{v} \times \mathbf{B}$$

- To fit within the magnetostatics framework, we must express this in terms of currents. We replace the one test charge with an infinitesimal length $d\mathbf{l}$ of current I :

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$$

- The total force on the wire is just the integral of the force on all the wire segments:

$$\mathbf{F} = \int I d\mathbf{l} \times \mathbf{B}$$

Magnetic Force on a Wire

- To make this more general and apply to any current density, we expand the current:

$$\mathbf{F} = \int \int \mathbf{J} \cdot \hat{\mathbf{l}} da d\mathbf{l} \times \mathbf{B}$$

- Combine the integral over the cross-sectional area with the line integral to get a volume integral:

$$\mathbf{F} = \int \mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x}) d\mathbf{x}$$

Magnetic Force on a Current Density

$$\mathbf{N} = \int \mathbf{x} \times (\mathbf{J}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})) d\mathbf{x}$$

Magnetic Torque on a Current Density

5. The Vector Potential:

- In electrostatics, the mathematics was simplified by introducing an electric potential and defining the electric field in terms of it. In a similar manner, let us define the magnetic field in terms of a vector potential:

$$\mathbf{B}(\mathbf{x}) = \nabla \times \mathbf{A}$$

Vector Potential Definition

- Previously we derived the Biot-Savart Law in curl form:

$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

$$\nabla \times \mathbf{A} = \frac{\mu_0}{4\pi} \nabla \times \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}' + \nabla \Psi$$

- The curl does not uniquely define the vector potential. We can always add the gradient of some scalar (called a gauge transformation) to the vector potential and end up with the same magnetic field. This underscores the fact the magnetic fields and forces are the real physical things and the vector potential is just a useful mathematical entity. If we choose the gauge where $\Psi = \text{constant}$, then we have:

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'$$

Biot-Savart Law for Vector Potential

- We can now express Ampere's Law in differential form in terms of the vector potential:

$$\nabla \times \mathbf{B}(\mathbf{x}) = \mu_0 \mathbf{J}(\mathbf{x})$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{J}(\mathbf{x})$$

- Use the vector identity: $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}(\mathbf{x})$$

- Choosing the gauge $\Psi = \text{constant}$ as was done above, leads to $\nabla \cdot \mathbf{A} = 0$, so that we have:

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}(\mathbf{x})$$

Ampere's Law in differential vector potential form