Lecture 1 Notes, Electromagnetic Theory I

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1. Classical Electromagnetism is a Macroscopic Theory

- Quantum theory is required to accurately describe atomic-scale electromagnetism.
- Classical electromagnetism is therefore a *macroscopic* theory. Classical magnetism is accurate, intuitive, and self-consistent as long as we restrict it to the macroscopic realm.
- There is no valid concept of an electron in classical electromagnetism. The word "electron" is hereby banned from this class. Instead we speak of an "electric charge" which is a large collection of electrons or other electrically-charged particles.
- For mathematical ease, we define point charges in classical electromagnetism. However, point charges do not represent electrons.
- A point charge q represents a macroscopic object that contains millions of electrons or other electrically-charged small particles, and yet is small enough compared to the rest of the system that the object can be approximated to be condensed to a single point in space.
- The line charge density λ is similarly an idealized mathematical construct consisting of charge lying along a wire with zero diameter. The line charge density of classical electromagnetic theory physically corresponds to a charged wire with a diameter much larger than an atom but much smaller than the rest of the system so that the diameter can be approximated to be zero.
- The surface charge density σ is an idealized mathematical construct consisting of charge lying along a two-dimensional surface with zero thickness. The surface charge density of classical electromagnetic theory physically corresponds to charged sheet with a thickness much larger than an atom but much smaller than the rest of the system so that the thickness can be approximated to be zero.
- The volume charge density ρ represents a collection of charge spread out through all three dimensions of space.

2. Idealized Zero-Width Objects Are Mathematically Represented Using Dirac Deltas

- Dirac delta definition: $\delta(x-x_0)=0$ for $x\neq x_0$ and $\int \delta(x-x_0) f(x)\,dx = f(x_0)$

- Multiple dimensions: $\delta^{(2)}(\mathbf{x})=\delta(x-x_0)\delta(y-y_0)$, $\delta^{(3)}(\mathbf{x})=\delta(x-x_0)\delta(y-y_0)\delta(z-z_0)$

- The superscript is usually omitted and the number of Dirac deltas involved is inferred from the context of the problem.

- Point charge: $\rho(\mathbf{x})=q_i\,\delta^{(3)}(\mathbf{x}-\mathbf{x}_i)$ where $\mathbf{x}_i$ is the position of the charge.

- Line charge: $\rho(\mathbf{x})=\lambda(\mathbf{x})\delta^{(2)}$

- Surface charge: $\rho(\mathbf{x})=\sigma(\mathbf{x})\delta^{(1)}$

- Dirac deltas obey the property: $\delta(au)=\frac{1}{|a|}\delta(u)$

- This identity shows us that the units of a Dirac delta are the inverse of the units of its operand. We could have guessed this fact by examining its integral definition. It is useful to check an equation by making sure the units work out, and the identity above facilitates such a units check.
- The general expression for the three-dimensional Dirac delta in any orthogonal coordinate system (u, v, w) with incremental length elements (du/U, dv/V, dw/W) is given by:

$$\delta(\mathbf{x}-\mathbf{x}_0)=U\,\delta(u-u_0)\,V\,\delta(v-v_0)\,W\,\delta(w-w_0)$$

- For example, in spherical coordinates, (r,θ,ϕ), the incremental length elements are $(dr, r\,d\theta, r\sin\theta\,d\phi)$. This means that $U=1$, $V=\frac{1}{r}$, and $W=\frac{1}{r\sin\theta}$, leading to:

$$\delta(\mathbf{x}-\mathbf{x}_0)=\delta(r-r_0)\frac{\delta(\theta-\theta_0)}{r}\frac{\delta(\phi-\phi_0)}{r\sin\theta}$$

- If we needed only a two dimensional delta in spherical coordinates we would omit the delta *and* length element for the dimension in which the charge distribution is not collapsed. For instance, a line charge shaped into a circular ring with radius R centered on the origin and lying in the x-y plane would have the equation:

$$\rho(r,\theta,\phi)=\lambda\,\delta(r-R)\frac{\delta(\theta-\pi/2)}{r}$$

- As another example, if charge is spread over a thin spherical shell, we only need one Dirac delta:

$$\rho(r,\theta,\phi)=\sigma\,\delta(r-R)$$

- Some complications arise if we try to represent charges at the origin or along the z axis. In such cases, we can find the right form (1) by ensuring the expression has the right units, and (2) by integrating over the charge density and ensuring that the result is the total charge. For instance, the correct expression for a point charge at the origin in spherical coordinates is:

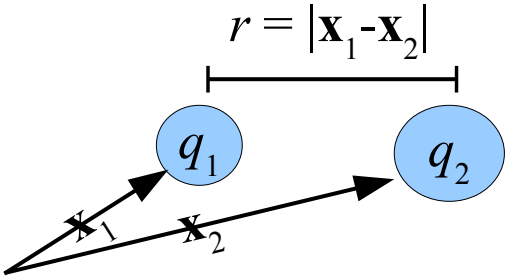
$$\rho(r,\theta,\phi)=q\frac{\delta(r)}{4\pi r^2}$$

- The factors in the denominator may seem like they do not belong there since the entire expression only contains a Dirac delta in the r direction and therefore we only need the incremental length factor $U = 1$. However, the factors in the denominator are required in order to ensure that the charge density has units of charge per length cubed and to ensure that the integral of this entire expression correctly gives the total charge of q.
- Similarly, the correct expression for a uniform line charge density along the z axis in cylindrical coordinates is:

$$\rho_c(\rho,\phi,z)=\lambda\frac{\delta(\rho)}{2\pi\rho}$$ (where ρ_c is the charge density and ρ is the radial coordinate.)

3. Coulomb's Law

- Coulomb found experimentally that for two electric point charges exerting a force on each other:
  1. force $\propto$ charge
  2. force $\propto$ 1/(distance^2)
  3. force is central
  4. force is attractive for oppositely charge bodies



- We can assemble each of these observations into an equation:

$$F \propto q_1 q_2 \;\rightarrow\; F \propto \frac{q_1 q_2}{r^2} \;\rightarrow\; \mathbf{F} \propto \frac{q_1 q_2}{r^2}\hat{r} \;\rightarrow\; \mathbf{F} = k\frac{q_1 q_2}{r^2}\hat{r}$$

- Convert this equation to a fixed coordinate system in order to be mathematically useful:

$$\mathbf{F} = k\frac{q_1 q_2}{|\mathbf{x}_1 - \mathbf{x}_2|^2}\hat{r}$$

$$\mathbf{F} = k\frac{q_1 q_2}{|\mathbf{x}_1 - \mathbf{x}_2|^2}\frac{(\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|}$$

$$\mathbf{F} = k\frac{q_1 q_2 (\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3}$$

- Be careful to not let the 3 in the exponent of the denominator to deceive you. The force still obeys an inverse square law because one of the powers of r in the denominator cancels the magnitude r in the numerator.
- The proportionality constant k is dependent on the units used and is found experimentally.

- For SI units, $k = \frac{1}{4\pi\epsilon_0}$ in vacuum.

- The final form becomes:

$$\boxed{\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2 (\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3}}$$

Coulomb's Law, in terms of the force, for two charges

- To be more general, factor out the one charge feeling the force and define the electric field **E**:

$$\mathbf{F} = q_1\left[\frac{1}{4\pi\epsilon_0}\frac{q_2 (\mathbf{x}_1 - \mathbf{x}_2)}{|\mathbf{x}_1 - \mathbf{x}_2|^3}\right]$$

$$\mathbf{F} = q_1\mathbf{E} \quad \text{where} \quad \mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\frac{q(\mathbf{x} - \mathbf{x}_1)}{|\mathbf{x} - \mathbf{x}_1|^3}$$

- The electric field is a vector field. It is defined at every point in space and can therefore be expressed as a function of spatial coordinates. Every point in space has an electric field vector with direction and magnitude.
- In contrast, the force is not a field of vectors, but is a single vector attached to the object feeling the force. We can make this more explicit by rewriting the force equation:

$\mathbf{F}_{\text{on } q_1} = q_1 \mathbf{E}(\mathbf{x}_1)$ where $\mathbf{x}_1$ is the location of charge q_1

- What if we want to know the force exerted on an extended charge distribution $\rho(\mathbf{x})$ when placed in an electric field $\mathbf{E}$? Then we must sum over all the infinitesimal forces felt by all the infinitesimal bits of charge that make up the extended object:

$$\mathbf{F} = \int d\,\mathbf{F}$$

$$\mathbf{F} = \int \mathbf{E}(\mathbf{x})\, d\,q$$

$$\mathbf{F} = \int \frac{d\,q}{d^3\mathbf{x}} \mathbf{E}(\mathbf{x}) d^3\mathbf{x}$$

$$\boxed{\mathbf{F} = \int \rho(\mathbf{x}) \mathbf{E}(\mathbf{x}) d^3\mathbf{x}}$$

- When using this equation, there is still just one net force vector $\mathbf{F}$ that the object $\rho(\mathbf{x})$ experiences.
- Now that we know how to handle forces exerted by fields, we can focus only on the electric field.
- Experimental observation shows that fields add linearly, therefore the total electric field is the vector sum of the field created by each point charge:

$$\boxed{\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_i q_i \frac{(\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^3}}$$ *Coulomb's Law, in terms of the field, for many charges*

- In the mathematical limit of increasingly smaller charges that are closer together, the sum becomes an integral over a continuous charge distribution:

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} dq$$

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \frac{d\,q}{d^3\mathbf{x}'} d^3\mathbf{x}'$$

$$\boxed{\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{(\mathbf{x} - \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|^3} \rho(\mathbf{x}')\, d\,\mathbf{x}'}$$ *Coulomb's Law, in terms of the field, for a charge density*

- We can use the above equation to find the electric field created by various idealized charge distributions:

- For a sheet charge, using $\rho(\mathbf{x}')=\sigma(\mathbf{x}')\delta^{(1)}$ gives $\mathbf{E}(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\sigma(\mathbf{x}')d^2\mathbf{x}'$

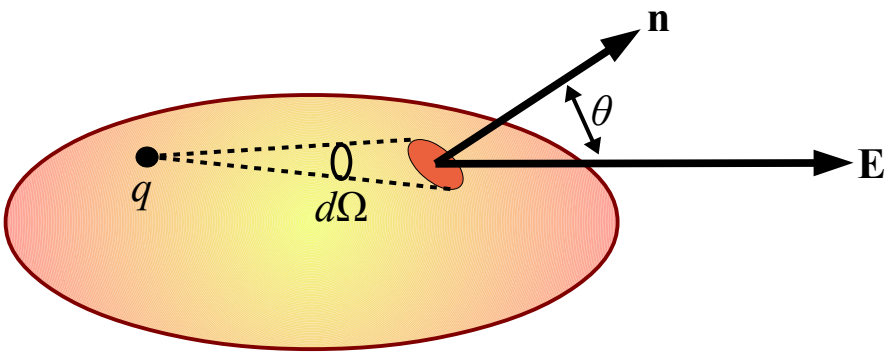
- For a line charge, using $\rho(\mathbf{x}')=\lambda(\mathbf{x}')\delta^{(2)}$ gives $\mathbf{E}(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\lambda(\mathbf{x}')d\mathbf{x}'$

- For a point charge, using $\rho(\mathbf{x}')=q\,\delta^{(3)}(\mathbf{x}'-\mathbf{x}_1)$ returns $\mathbf{E}(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\frac{(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3}q$

- Coulomb's Law does not include bounding surfaces. Therefore the integral must be done over all charges in the universe, or we must assume that most of the charges in the universe are far enough away that their contribution is negligible. Because of this fact, Coulomb's law has limited applicability. We can cast it into forms that are more useful.

4. Gauss's Law

- Take a point charge q and fix a simple, closed mathematical surface S around it (called a Gaussian surface).
- At some point $\mathbf{x}$ on the surface, there is a vector $\mathbf{n}$ normal to the surface and an electric field vector $\mathbf{E}$ arising from the point charge q.



- Take Coulomb's law for a point charge: $\mathbf{E}(\mathbf{x})=\frac{1}{4\pi\epsilon_0}q\frac{(\mathbf{x}-\mathbf{x}_1)}{|\mathbf{x}-\mathbf{x}_1|^3}$

- Put the charge at the origin: $\mathbf{E}=\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$

- Dot both sides by the normal vector: $\mathbf{E}\cdot\mathbf{n}=\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta$

- Integrate both sides over the closed surface: $\oint_S\mathbf{E}\cdot\mathbf{n}\,da=\oint_S\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta\,da$

- Expand the surface element using $\cos\theta\,da=r^2\,d\Omega$:

$$\oint_S\mathbf{E}\cdot\mathbf{n}\,da=\oint_S\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}r^2\,d\Omega$$

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{4\pi\epsilon_0} q \oint_S d\Omega$$

$$\boxed{\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{\epsilon_0}}$$ where q is the total charge enclosed by the Gaussian surface.

- Even though we derived the equation above using a point charge, it is valid for any charge distribution with total charge q completely enclosed by the Gaussian surface.
- Expand the total enclosed charge in the equation above into an integral over the charge distribution filling the volume V enclosed by surface S.

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\epsilon_0}\int_V d\,q$$

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\epsilon_0}\int_V \frac{d\,q}{d^3\mathbf{x}} d^3\mathbf{x}$$

$$\boxed{\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}) d^3\mathbf{x}}$$ *Gauss's Law in integral form for a charge distribution*

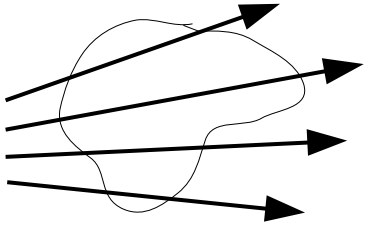
- The integral form of Gauss's Law allows us to know the total electric field flux through a closed surface if we know the enclosed charge. If the electric field is constant across the surface and everywhere perpendicular to the surface, $\mathbf{E} = E_0\mathbf{n}$, the electric field can be brought out of the integral and can therefore be determined:

$$E_0 \oint_S da = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x}) d^3\mathbf{x}$$

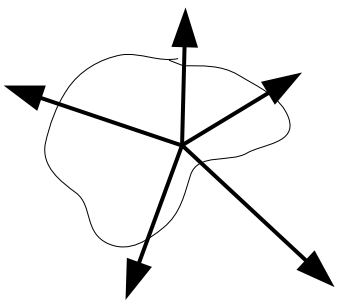
$$E_0 = \frac{1}{\epsilon_0 A_S}\int_V \rho(\mathbf{x}) d^3\mathbf{x}$$ where A_s is the area of the surface S

- Note that this equation is only useful in very simple cases where a Gaussian surface can be drawn for which the electric field is constant and perpendicular to the surface.
- The divergence theorem states that for any vector field **A** the total flux of **A** through the closed surface S equals the integral over the volume V of the divergence of **A**, where V is the volume bounded by the surface S:

$$\oint_S \mathbf{A}\cdot\mathbf{n}\,da = \int_V \nabla\cdot\mathbf{A}\, d^3\mathbf{x}$$



NO DIVERGENCE



HIGH DIVERGENCE

- Apply the divergence theorem to the left side of Gauss's law in integral form to find:

$$\int_V \nabla \cdot \mathbf{E}\, d^3\mathbf{x} = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{x})\, d^3\mathbf{x}$$

- Because this must be true for any arbitrary volume over which the integral is done, we can shrink the volume down repeatedly to every point in space. Thus the integrands must be equal, leading to:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho(\mathbf{x})}$$ *Gauss's Law in differential form*

- As opposed to Coulomb's law and Gauss's law in integral form, Gauss's law in differential form is a local equation. It connects the charge density at one point and the electric field at the same point. For this reason, Gauss's law in differential form is often the most useful in practice.
- According to Gauss's Law, if there is no charge present in a certain region, all the electric field lines that enter this region must also exit that region. In other words, Gauss's law tells us that positive electric charge creates electric field lines and negative electric charge destroys field lines. If there is no charge in a certain region, electric field lines are not created or destroyed in that region.

5. The Scalar Potential

- Gauss's law involves the electric field, which is a vector field. Because solving vector equations is much more difficult than solving scalar equations, the mathematics can be simplified by transforming Gauss's law into a scalar form.
- Electrostatic fields are experimentally observed to be irrotational (have no curl):

$$\boxed{\nabla \times \mathbf{E} = 0}$$ *Electrostatic fields are irrotational*

- Interestingly, mathematics tells us that the gradient of any scalar field also has no curl:

$$\nabla \times (\nabla \Phi) = 0$$

- Comparing the two equations above, we can define the electric field in terms of some electrostatic scalar potential:

$$\boxed{\mathbf{E} = -\nabla \Phi}$$

- Mathematically, we could have chosen the sign in this definition to be positive or negative. We choose negative so that the scalar potential will relate to potential energy in a straight-forward way. Mathematically, the gradient operator points uphill in the scalar function landscape and the negative gradient operator points downhill. Since balls roll downhill, and electric charges feel a force downhill in the potential energy landscape, we choose the negative gradient operator.
- It should be noted that in a strict sense, the scalar potential itself is just a mathematical entity which simplifies calculations, and has no formal physical meaning. (The Aharonov–Bohm effect seems to dictate that the potentials have physical meaning. But the Aharonov–Bohm effect is a quantum effect. In the self-consistent realm of classical electromagnetidm, the potentials have no physical meaning.) However, *differences* of the scalar potential (also known

as potential differences or voltages) can be *related* to things with physical meaning.
- The work done in moving a charge from point A to point B in the presence of an electric field is just the integrated force times distance:

$$W = -\int_A^B \mathbf{F} \cdot d\,\mathbf{l}$$

- Substitute into the above equation the force **F** in terms of the electric field, and the electric field in terms of the potential:

$$W = -\int_A^B (q_1 \mathbf{E}) \cdot d\,\mathbf{l} \quad \rightarrow \quad W = q_1 \int_A^B \nabla \Phi \cdot d\,\mathbf{l} \quad \rightarrow \quad W = q_1 [\Phi(\mathbf{x}_B) - \Phi(\mathbf{x}_A)] \text{ or } W = q_1 \Delta\Phi$$

- Therefore, the potential difference between two points is the work done W to move a test charge q_1 between these two points divided by q_1: $\Delta\Phi = W / q_1$.
- By the work-energy theorem, the potential difference can also be interpreted as the potential energy U of a test charge q_1 divided by q_1:

$$\boxed{\Delta\Phi = U / q_1}$$

- As an analogy, electrostatics can be thought of as balls on a hill. Pushing a ball up a hill takes work and gives the ball gravitational potential energy. Similarly, pushing a positive point charge to points with higher electric scalar potential values (such as towards another positive point charge) takes work and gives the charge electrostatic potential energy. A ball on the side of the hill feels a net force that is in the direction of steepest descent. Similarly, an electric charge in an electric field feels a force that is in the direction that the scalar potential decreases the fastest.
- Next note that *the electrostatic scalar potential is always continuous* (except across line charges and dipole charge layers – but these are unphysical idealizations).
- The continuous nature of the electric potential can be seem from the fact that a discontinuous potential would lead to an infinite slope, and therefore an infinite electric field. An infinite force would be required to push a test charge past the discontinuity.
- The continuity of the electric potential can be used as a boundary condition on the potential but not on the fields.
- The relationship between the scalar potential Φ and the potential energy U is similar to the relationship between the electric field **E** and the force **F**. The scalar potential is a scalar field that exists at every point in space. In contrast, the potential energy is a single number attached to a specific object that is experiencing the scalar potential at a certain point in space. We can make this more explicit by adding more labels to the equation:

$U_{\text{of } q_1} = q_1 [\Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_{\text{ground}})]$ where $\mathbf{x}_1$ is the location of charge q_1

or

$U_{\text{of } q_1} = q_1 \Phi(\mathbf{x}_1)$ if the ground location is chosen such that $\Phi(\mathbf{x}_{\text{ground}}) = 0$

- We can now transform the electrostatic equations into forms that are in terms of the scalar potential.

- Take Coulomb's Law and use the scalar potential definition:

$$\mathbf{E}(\mathbf{x})=\frac{1}{4\pi\epsilon_0}\int\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$-\nabla\Phi=\frac{1}{4\pi\epsilon_0}\int\frac{(\mathbf{x}-\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|^3}\rho(\mathbf{x}')d\,\mathbf{x}'$$

- Now use the identity: $\frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3}=-\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}$ (Prove this for yourself and save for later.)

$$-\nabla\Phi=\frac{1}{4\pi\epsilon_0}\int\left[-\nabla\frac{1}{|\mathbf{x}-\mathbf{x}'|}\right]\rho(\mathbf{x}')d\,\mathbf{x}'$$

$$\boxed{\Phi=\frac{1}{4\pi\epsilon_0}\int\frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\,\mathbf{x}'+C}$$ *Coulomb's law in terms of the scalar potential*

- The arbitrary constant is usually made to go away by defining a ground and specifying the potential as zero at the ground.
- Take Gauss's law in differential form and use the scalar potential definition:

$$\nabla\cdot(-\nabla\Phi)=\frac{1}{\epsilon_0}\rho(\mathbf{x})$$

$$\boxed{\nabla^2\Phi=-\frac{1}{\epsilon_0}\rho(\mathbf{x})}$$ *Poisson Equation*

- A special case of the Poisson equation results when a region has no charges:

$$\boxed{\nabla^2\Phi=0}$$ *The Laplace Equation*

- The potential in the Laplace Equation is defined uniquely by the boundary conditions alone.

6. Capacitance

- A special class of systems in electrostatics involves a collection of separate objects that are perfect conductors residing in free space.
- Because such systems have only conductors and free space, the only properties involved are the charges, the conductor potentials and geometric properties linking the two.
- The electrostatic potential at any point in space due to a collection of *n* separate objects is just the sum over the potentials due to each object:

$$\Phi=\frac{1}{4\pi\epsilon_0}\sum_{j=1}^{n\ \text{objects}}\int\frac{\rho_j(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d\,\mathbf{x}'$$

- This expression simplifies if we only care about points in space on the surface of one of the objects. The surface of each conductor is a constant value (an equipotential) V_i:

$$V_i = \frac{1}{4\pi\epsilon_0} \sum_{j=1}^{n\ \text{objects}} \int \frac{\rho_j(\mathbf{x}')}{|\mathbf{x}_i - \mathbf{x}'|} d\,\mathbf{x}'$$ where $\mathbf{x}_i$ is a point on the surface of object i.

- When dealing with conductors, the charge density tends to spread out into the same minimal-energy pattern every time no matter how much charge is applied. For this reason, the total charge Q_j on the j^{th} object is an externally applied property, but the normalized charged density ρ_j/Q_j is a function of the system's geometry. We factor out Q_j explicitly:

$$V_i = \sum_{j=1}^{n\ \text{objects}} \left[\frac{1}{4\pi\epsilon_0} \int \frac{\rho_j(\mathbf{x}')/Q_j}{|\mathbf{x}_i - \mathbf{x}'|} d\,\mathbf{x}' \right] Q_j$$

- At this point, the quantity in brackets is purely a function of system geometry, which does not change. We can calculate it once for a certain system by using a dummy set of charges and potentials, and then know this number forevermore.

$$V_i = \sum_{j=1}^{n\ \text{objects}} p_{ij} Q_j \quad \text{where} \quad p_{ij} = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_j(\mathbf{x}')/Q_j}{|\mathbf{x}_i - \mathbf{x}'|} d\,\mathbf{x}'$$

- The variable p_{ij} is the normalized electric potential felt at the i^{th} conductor due to the j^{th} conductor.
- This tells us that once we know the total charge we applied to each conductor and their geometrical coefficients p_{ij}, we can find their potentials.
- The opposite case is more useful in practice: we apply certain potentials to each conductor, and we want to know the total charges.
- Fortunately, the linear system of n equations can be inverted to yield:

$$Q_i = \sum_{j=1}^{n\ \text{objects}} C_{ij} V_j$$

- The variable C_{ii} is the capacitance of the i^{th} object, and the variables C_{ij}, $i \neq j$ are the coefficients of inductance.
- Because we are dealing with perfect conductors, the capacitances and coefficients of inductance do not depend on material properties. They depend only on the geometry (shape and relative location) of the objects.
- For example, suppose we have only two conductors (two parallel plates or two concentric cylinders) and we want to calculate the capacitance so that we can stamp it on the side of the packaging for future reference. The general equations are:

$$Q_1 = C_{11} V_1 + C_{12} V_2$$
$$Q_2 = C_{21} V_1 + C_{22} V_2$$

- Let us focus on the first equation. We are always free to add or subtract an overall constant to all the potentials without changing the physics. Let us subtract V_2 from each potential to get rid of the second term.

$$Q_1 = C_{11}(V_1 - V_2)$$

- Now solve for the capacitance:

$$C_{11} = \frac{Q_1}{(V_1 - V_2)}$$

- For this simple system, there is a symmetry so that if both objects have equal and opposite charge, they will have the same capacitance, so that we can write:

$$C = \frac{Q}{\Delta V}$$ *Capacitance of Two-Conductor System*

- Because Q is the total charge on one conductor and ΔV is the potential difference between the two conductors, we can interpret the capacitance as a *geometry-dependent quantity that describes the system's ability to hold charge*.
- A system with a higher capacitance can hold more charge for a given potential difference.
- If holding as much charge as possible is the aim of a capacitor, its geometry should be altered to maximize the capacitance.