Lecture 8 Notes, Electromagnetic Theory II

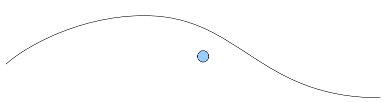
Dr. Christopher S. Baird, faculty.uml.edu/cbaird
University of Massachusetts Lowell

C. S. BAIRD

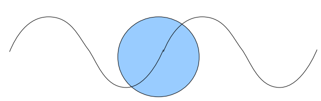
1. Scattering Introduction

- Consider a localized object that contains charges that are effectively static. Left to itself, it creates no significant electric or magnetic fields in the far field.
- An electromagnetic wave propagates until it reaches the object. Part of the wave may pass by unaffected.
- The fields in the wave exert forces on the electric charges in the object and induce oscillations.
- The oscillating charges in turn radiate waves that propagate radially away, generally in every direction.
- The incident wave is therefore scattered by the object.
- Classically, electromagnetic scattering can be broken down into three general categories based on how the wavelength λ of the electromagnetic wave compares to the diameter d of the object:

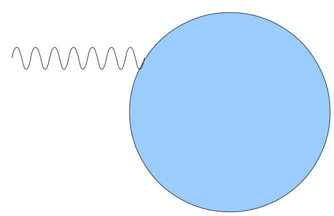
Long-Wavelength Scattering
a.k.a. "Rayleigh Scattering"
$\lambda >> d$
pseudo-static scattering dominates



Mid-Wavelength Scattering
a.k.a. "Resonant Scattering"
For spheres: "Mie Scattering"
$\lambda \approx d$
interference effects dominate



Short-Wavelength Scattering
a.k.a. "Ray Optics"
a.k.a. "Geometric Optics"
$\lambda << d$
reflection, refraction, and dispersion dominate



2. Short-Wavelength Scattering (Ray Optics)

- In the short-wavelength category, the electromagnetic waves act somewhat like little rays of light that travel and reflect like marbles.
- Rather than solving Maxwell's equations in their complete form, we can mathematically approximate the waves to be little rays.
- Because $\lambda << d$, the scattering object is effectively flat for a given ray. Therefore, all we need to do to calculate the scattering is to shoot a grid of rays at the object. For each ray, we simply apply the laws of refraction and reflection at the point where the ray strikes the object's surface and then follow the transmitted and reflected rays until they hit another surface or escape to infinity. The process is called "ray-tracing".
- Since most everyday objects are much larger than the wavelengths of visible light, short-wavelength scattering is the type of scattering that we are most familiar with in everyday life. For this reason, computational engines that are used to create 3D computer animated movies and video games are mainly ray-tracing engines that use ray optics algorithms.
- The behavior of traditional optical lenses and mirrors falls into the category of short-wavelength scattering, and should have been extensively covered in your undergraduate classes.

- The ray optics approximation ignores wave effects such as polarization, interference, and diffraction, and therefore has limited accuracy.
- Even though dispersion is fundamentally a wave effect, we can include dispersion in a ray optics model simply by making the index of refraction frequency-dependent. For instance, the way in which raindrops create rainbows through dispersion can be accurately described using only ray optics and the experimentally-measured frequency dependence of water's index of refraction.
- Since we have already covered reflection, refraction, and dispersion in this class, we have effectively already covered short-wavelength scattering. We will therefore say no more about this category.
- There exists another category of scattering called "Physical Optics". The name of this model is vague and misleading. Physical Optics is an approximation to Maxwell's equations that starts with the ray optics approximation and then manually adds on a few diffraction, interference, and/or polarization effects. There are many models that are Physical Optics models since there are many different wave effects that can be manually added on, and there are many different ways to apply approximations to the wave effects. Maxwell's equations constitute the only complete, exact, self-consistent description of classical electromagnetic scattering. The various Physical Optics models are approximations to Maxwell's equations.

3. Mid-Wavelength Scattering (Resonant Scattering)

- In the mid-wavelength category we generally can't make any approximations and we have to solve Maxwell's equations in their complete form.
- Usually, the only way to solve Maxwell's equations in their complete form for complicated scattering objects is using numerical algorithms. Methods that solve Maxwell's equations for scattering problems without making any approximations are called "Full-Wave" algorithms.
- For mid-wavelength scattering, resonance and interference effects tend to dominate.
- For the special case of mid-wavelength scattering off of spheres, Maxwell's equations can be approached analytically without needing to make approximations. The solution is called "Mie Scattering".

4. Long-Wavelength Scattering (Rayleigh Scattering)

- If the wavelength of the incident light is very long compared to the size of the scattering object, we can use the concepts we developed when describing radiation; the object can be described by the lowest multipoles, which oscillate in step with the incident wave.
- The oscillating electric and magnetic multipoles of a scattering object are induced by the incident electromagnetic wave.
- Since the wavelength is so large compared to the scattering object, we can approximate the electric and magnetic field as uniform across the object at an instant in time. Therefore, we just need to find the electric and magnetic multipoles induced in the object by a static, uniform electric and magnetic field, and then find the radiation emitted by such multipoles when they are oscillating.
- Static uniform fields dominantly induce dipole moments, so that is what we will focus on here.
- Consider an object in free space sitting at the origin.
- A plane monochromatic wave of polarization $\boldsymbol{\epsilon}_0$ traveling in the z direction is incident on the object:

$$\mathbf{E}_{\text{inc}} = \boldsymbol{\epsilon}_0 E_0 e^{i(k_0 z - \omega t)} \quad , \quad \mathbf{B}_{\text{inc}} = \frac{1}{c}(\mathbf{k}_0 \times \boldsymbol{\epsilon}_0) E_0 e^{i(k_0 z - \omega t)}$$

- These fields induce an electric dipole **p** and magnetic dipole **m** in the object.
- The scattered waves have an electric field that are equal to the far-field radiation created by oscillating dipoles, which were found in a previous lecture.

$$\mathbf{E}_{\mathrm{sc}} = -\frac{k^2}{4\pi\epsilon_0}\left[\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p}) + \frac{1}{c}\hat{\mathbf{k}}\times\mathbf{m}\right]\frac{e^{i(kr-\omega t)}}{r}$$

- The scattering effects are typically measured experimentally as a *differential scattering cross section*. This is defined as the power radiated with polarization $\boldsymbol{\epsilon}$, per unit solid angle, per unit incident flux:

$$\frac{d\sigma}{d\Omega} = \frac{r^2\left|\boldsymbol{\epsilon}^*\cdot\mathbf{E}_{\mathrm{sc}}\right|^2}{\left|\boldsymbol{\epsilon}_0^*\cdot\mathbf{E}_{\mathrm{inc}}\right|^2}$$ *Polarization-Specific Differential Scattering Cross Section*

- The complex-conjugate operators are present in this equation to make the circular-polarization basis vectors work properly. Since the linear-polarization basis vectors are real-valued, the complex conjugation goes away for the case of a linear-polarization basis.
- We plug in our incident and scattered fields to find:

$$\frac{d\sigma}{d\Omega} = \frac{r^2\left|\boldsymbol{\epsilon}^*\cdot\left(-\frac{k^2}{4\pi\epsilon_0}\left[\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p}) + \frac{1}{c}\hat{\mathbf{k}}\times\mathbf{m}\right]\frac{e^{i(kr-\omega t)}}{r}\right)\right|^2}{\left|E_0\right|^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\epsilon}^*\cdot\left[\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p}) + \frac{1}{c}\hat{\mathbf{k}}\times\mathbf{m}\right]\right|^2$$

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\epsilon}^*\cdot(\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{p})) + \frac{1}{c}\boldsymbol{\epsilon}^*\cdot(\hat{\mathbf{k}}\times\mathbf{m})\right|^2$$

- We can make this equation more useful by moving the measured polarization vectors inside the parentheses using the vector identities $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ and $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\epsilon}^*\cdot(\hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\mathbf{p}) - \mathbf{p}) - \frac{1}{c}(\hat{\mathbf{k}}\times\boldsymbol{\epsilon}^*)\cdot\mathbf{m}\right|^2$$

- The polarization vector is always perpendicular to the propagation vector, so the first term vanishes, leaving:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\epsilon}^*\cdot\mathbf{p} + \frac{1}{c}(\hat{\mathbf{k}}\times\boldsymbol{\epsilon}^*)\cdot\mathbf{m}\right|^2}$$

- This is the simplest we can get the equation without knowing the dipole moments.
- To solve a specific problem, we would place the scattering object in a static uniform electric field and a static uniform magnetic field, calculate the induced electric dipole and magnetic

dipole moments in the object by the uniform fields, and plug the dipoles into the above equation.

5. Scattering by a Small Dielectric Sphere

- Consider a non-magnetic sphere of radius *a* made out of uniform linear dielectric sitting in free space.
- Since the material is not magnetic, there is no magnetic dipole moment, $\mathbf{m} = 0$.
- The electric dipole moment induced in a dielectric sphere by a uniform electric field was found last semester when we were learning electrostatics:

$$\mathbf{p} = 4\pi\epsilon_0\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)a^3\,\mathbf{E}_{\mathrm{inc}}(0)$$

- Writing out the polarization unit vector explicitly, this becomes:

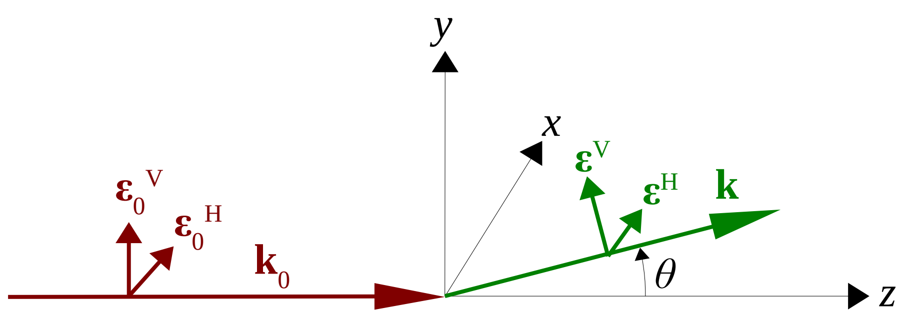
$$\mathbf{p} = 4\pi\epsilon_0\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)a^3\,\boldsymbol{\epsilon}_0 E_0$$

- Plugging these dipole moments into the differential cross section equation leads to:

$$\frac{d\sigma}{d\Omega} = \frac{k^4}{(4\pi\epsilon_0 E_0)^2}\left|\boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon}_0\, 4\pi\epsilon_0\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)a^3 E_0\right|^2$$

$$\boxed{\frac{d\sigma}{d\Omega} = k^4 a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\left|\boldsymbol{\epsilon}^*\cdot\boldsymbol{\epsilon}_0\right|^2}$$

- This is a measure of the power scattered into some specific polarization $\boldsymbol{\epsilon}$ if some polarization $\boldsymbol{\epsilon}_0$ is incident.
- To get a better understanding what this means, let us pick a definite coordinate system and expand.
- The polarization vector lies in a plane perpendicular to the direction of propagation. We can thus span all possible polarizations for a single wave by picking two orthogonal polarization vectors. Let us call them horizontal (H) and vertical (V) and define them as in this diagram:



- The scattering plane is the plane containing both the incident and the scattered wavevectors.
- The horizontal polarization basis vector is normal to the scattering plane.
- The vertical polarization basis vector is parallel to the scattering plane.
- The scattering angle θ is the angle between the incident and scattered wavevectors.
- With two possible incident polarizations, and two possible scattered polarizations, there are four possible polarization-specific differential scattering cross sections:
- Applying the coordinate system defined above to our dielectric sphere, we find the four polarization-dependent cross sections to be:

$$\frac{d\sigma_{\mathrm{HH}}}{d\Omega}=k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2$$

$$\frac{d\sigma_{\mathrm{HV}}}{d\Omega}=0$$

$$\frac{d\sigma_{\mathrm{VH}}}{d\Omega}=0$$

$$\frac{d\sigma_{\mathrm{VV}}}{d\Omega}=k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\cos^2\theta$$

- The first subscript labels the polarization of the incident wave and the second subscript labels the polarization of the scattered wave. For instance, $\frac{d\sigma_{\mathrm{HV}}}{d\Omega}$ refers to horizontally-polarized incident waves and vertically-polarized scattered waves.
- As we see above, the dielectric sphere is so simple that it does not scatter waves into the cross-pol channels (HV and VH). In general, azimuthally-symmetric objects made out of isotropic, homogenous materials do not scatter waves into the cross-pol channels. More complicated objects do scatter waves into the cross-pol channels.
- In general, objects tend to scatter most strongly in the co-pols (HH and VV) and weakest in the cross-pols.
- Much information can be gained about a particular object by measuring all four polarization-dependent scattering cross sections. Such data are called “full-polarimetric measurements”.
- The field of study involving such measurements is called “polarimetry.”
- Measuring all four polarization channels improves the analysis of a scattering object. For instance, the differentiation of enemy targets is improved when full-polarimetric radar scattering measurements are taken. Also, polarimetric aerial images of land to be cultivated can be used to determine the water content in the ground.
- Sometimes it is more useful to ask the question, what is the pattern of scattered power if the incident wave is unpolarized?
- An unpolarized wave is better understood as a “randomly polarized” wave because polarization is just the electric field vector after the oscillations in time and space have been factored out, and the electric field vector must always have some direction.
- We can measure the scattering effects corresponding to unpolarized incident light by averaging the effects of two orthogonal incident incoherent waves:

$$\frac{d\sigma_{\text{unpol, H}}}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma_{\text{HH}}}{d\Omega} + \frac{d\sigma_{\text{VH}}}{d\Omega}\right]$$

$$\frac{d\sigma_{\text{unpol, V}}}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma_{\text{HV}}}{d\Omega} + \frac{d\sigma_{\text{VV}}}{d\Omega}\right]$$

- For the dielectric sphere, these become:

$$\boxed{\frac{d\sigma_{\text{unpol, H}}}{d\Omega} = \frac{1}{2} k^4 a^6 \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)^2}$$

$$\boxed{\frac{d\sigma_{\text{unpol, V}}}{d\Omega} = \frac{1}{2} k^4 a^6 \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)^2 \cos^2\theta}$$

- We can go one step further and ask, what is the total scattered power in a certain direction if there is unpolarized incident light?
- To find this we just sum the two possible scattered polarizations:

$$\frac{d\sigma_{\text{unpol, unpol}}}{d\Omega} = \frac{d\sigma_{\text{unpol, H}}}{d\Omega} + \frac{d\sigma_{\text{unpol, V}}}{d\Omega}$$

- This is equivalent to half of the sum of all the polarization channels:

$$\frac{d\sigma_{\text{unpol, unpol}}}{d\Omega} = \frac{1}{2}\left[\frac{d\sigma_{\text{HH}}}{d\Omega} + \frac{d\sigma_{\text{VH}}}{d\Omega} + \frac{d\sigma_{\text{HV}}}{d\Omega} + \frac{d\sigma_{\text{VV}}}{d\Omega}\right]$$

- For the dielectric sphere, the total scattered power for unpolarized incident light is:

$$\boxed{\frac{d\sigma_{\text{unpol, unpol}}}{d\Omega} = k^4 a^6 \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0}\right)^2 \frac{1}{2}\left(1 + \cos^2\theta\right)}$$

- Another interesting question we can ask is, if unpolarized light is incident, to what extent is the scattered light polarized?
- The "average polarization" $\Pi(\theta)$ of the scattered waves is measured as the difference of the two scattered polarizations over the total scattered power:

$$\Pi(\theta) = \frac{\dfrac{d\sigma_{\text{unpol, H}}}{d\Omega} - \dfrac{d\sigma_{\text{unpol, V}}}{d\Omega}}{\dfrac{d\sigma_{\text{unpol, unpol}}}{d\Omega}}$$

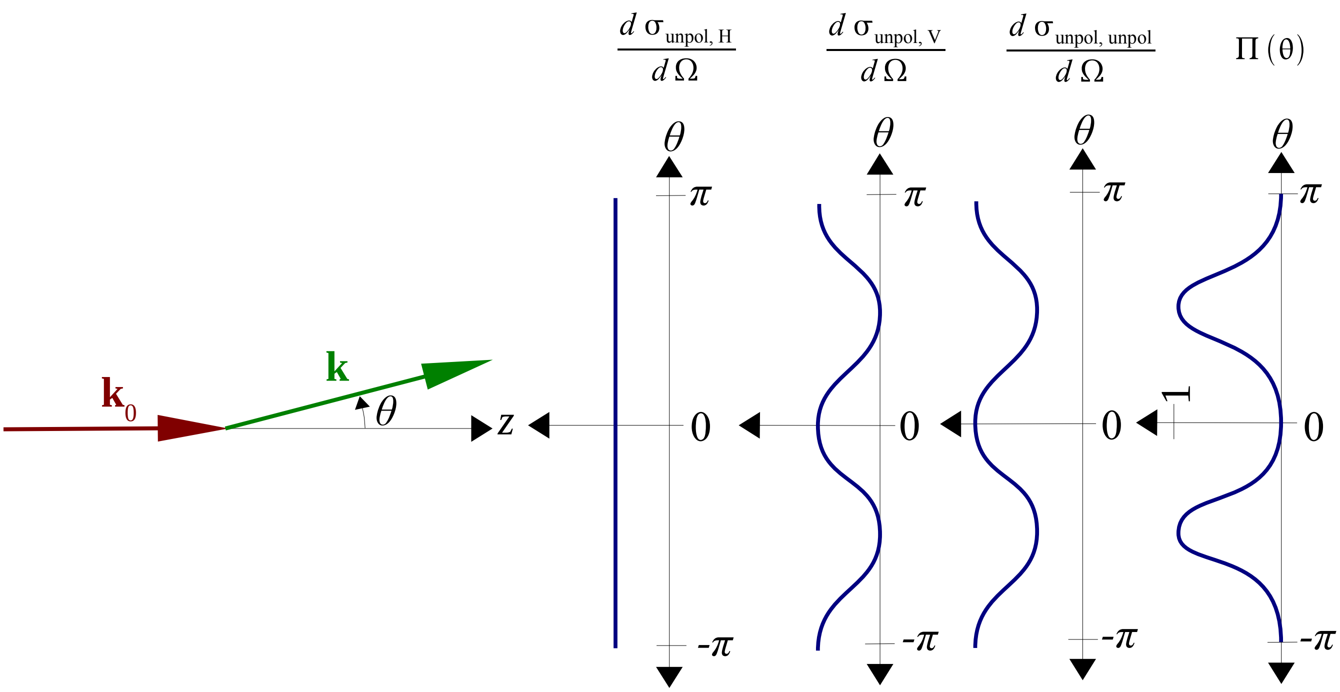
- We can see that the average polarization is a dimensionless number that ranges from -1 to 1.
- A value of 1 means that the scattered light in that direction is 100% horizontally polarized.
- A value of -1 means that the scattered light in that direction is 100% vertically polarized.

- A value of 0 means that the scattered light in that direction is unpolarized.
- Let us calculate the average polarization of the light scattered by a dielectric sphere:

$$\Pi(\theta)=\frac{\frac{1}{2}k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2-\frac{1}{2}k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\cos^2\theta}{k^4a^6\left(\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}\right)^2\frac{1}{2}(1+\cos^2\theta)}$$

$$\boxed{\Pi(\theta)=\frac{\sin^2\theta}{1+\cos^2\theta}}$$

- We can gain better understanding by plotting all of these parameters:



6. Scattering Matrix

- What if we want to preserve phase information? Then we must describe scattering in terms of parameters that link the incident and scattered electric fields rather than link the incident and scattered power.
- In general, a scattering object is an operator S that takes the incident electric field and transforms it into the scattered electric field:

$$\mathbf{E}_{\text{scat}}=S\,\mathbf{E}_{\text{inc}}$$

- To be as general as possible, we realize that the incident and scattered wave's electric field vector could each have two components. This requires the operator S, if we don't want it to contain vector operators, to be a two-by-two matrix:

$$\begin{bmatrix} E_H e^{i\theta_H} \\ E_V e^{i\theta_V} \end{bmatrix}_{\text{scat}} = \begin{bmatrix} S_{\text{HH}}(r) & S_{\text{VH}}(r) \\ S_{\text{HV}}(r) & S_{\text{VV}}(r) \end{bmatrix} \begin{bmatrix} E_H e^{i\theta_H} \\ E_V e^{i\theta_V} \end{bmatrix}_{\text{inc}}$$

- Note that the scattering matrix approach can be used for any type of scattering, and not just for long-wavelength scattering.
- In the far field, the scattered wave is always a spherical wave. We can factor out and ignore the radial dependence of the scattered field because we already know what it should look like for a spherical wave. In practice, this is accomplished by calibrating the system to a known scatterer at a known distance.

$$\begin{bmatrix} E_H e^{i\theta_H} \\ E_V e^{i\theta_V} \end{bmatrix}_{\text{scat}} = \frac{e^{ikr}}{r} \begin{bmatrix} S_{\text{HH}} & S_{\text{VH}} \\ S_{\text{HV}} & S_{\text{VV}} \end{bmatrix} \begin{bmatrix} E_H e^{i\theta_H} \\ E_V e^{i\theta_V} \end{bmatrix}_{\text{inc}}$$

which upon calibration becomes:

$$\begin{bmatrix} E_H e^{i\theta_H} \\ E_V e^{i\theta_V} \end{bmatrix}_{\text{scat}} = \begin{bmatrix} S_{\text{HH}} & S_{\text{VH}} \\ S_{\text{HV}} & S_{\text{VV}} \end{bmatrix} \begin{bmatrix} E_H e^{i\theta_H} \\ E_V e^{i\theta_V} \end{bmatrix}_{\text{inc}}$$

- It should be noted that we must use the scattering matrix if we care about phase information.
- The differential scattering cross sections are defined in terms of intensity, which depends on the magnitude-square of the electric fields, and thus phase information is lost when using differential scattering cross sections.
- Because the light's intensity is proportional to the magnitude-square of the electric field, the differential scattering cross sections are equal to the magnitude-square of the scattering matrix elements:

$$\frac{d\sigma_{\text{HV}}}{d\Omega} = |S_{HV}|^2 \quad \text{etc.}$$

- The scattering matrix approach is very useful because if the wave scatters off scatterer 1, and then scatterer 2, and then scatterer 3, the resultant wave can be found simply by multiplying scattering matrices:

$$\mathbf{E}_{\text{scat}} = S_3 S_2 S_1 \mathbf{E}_{\text{inc}}$$

- Note that when doing matrix multiplication, the order of the matrices matters. The scatterer that the incident wave encounters first should have its scattering matrix closest in the equation to the incident electric field vector.
- For the small dielectric sphere, we can immediately write down its scattering matrix (the spherical wave propagation factor is understood and omitted):

$$S = k^2 a^3 \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right) \begin{bmatrix} 1 & 0 \\ 0 & \cos\theta \end{bmatrix}$$

- This approach so far is very general. Typically, when a more specific approach is chosen, more

definitions are applied.
- For instance, in laser optics, we typically only care about the laser light that goes through the object and comes out the other side, known as the “forward-scattered wave”. The scattering matrix S becomes defined as the the Jones matrix J of laser optics when we set $\theta = 0$.
- With this definition, we can immediately write down the Jones matrix for several scatterers:

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$ Perfect Horizontal Polarizer

$$J = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$ Perfect Vertical Polarizer

$$J = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ Linear Polarizer at 45°

$$J = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ Linear Polarizer at -45°

$$J = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$ Linear Polarizer with its transmission axis at an angle θ with H

$$J = \frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$ Right Circular Polarizer

$$J = \frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$ Left Circular Polarizer

$$J = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$ Quarter-Wave Plate with fast axis horizontal (converts linear pol to circular pol)

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}$$ Quarter-Wave Plate with fast axis vertical (converts linear pol to circular pol)

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ Half-wave plate with fast axis horizontal or vertical (flips linear pol direction)

$$J = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\,d\,\Delta n\,k_0} \end{bmatrix}$$ Phase retarder plate with fast axis horizontal

$$J = \begin{bmatrix} 1 & 0 \\ 0 & e^{-i\,d\,\Delta n\,k_0} \end{bmatrix}$$ Phase retarder plate with fast axis vertical

- Here, d is the physical width of the plate, Δn is the birefringence of the plate, and k_0 is the free space wavenumber of the wave passing through the plate.
- Note that in each of the Jones matrices listed above an overall phase factor is neglected.

- As a separate application, in radar detection, the same antenna creates the incident wave and measures the scattered wave. Such a system thus measures the "back-scattered wave". The scattering matrix used in radar polarimetry is found by taking the general scattering matrix and setting $\theta = \pi$. Note that the coordinate system used in radar polarimetry is slightly different than the coordinate system used in optical polarimetry, so you cannot directly equate scattering matrices for a given object across both fields of study.