Lecture 7 Notes, Electromagnetic Theory II

Dr. Christopher S. Baird, faculty.uml.edu/cbaird
University of Massachusetts Lowell

1. Far-Field Dipole Radiated Power

- Using the Green's function solution for radiation, we applied it to a localized source.
- To find the far field behavior of the waves, we then expanded the exponential and kept only the first term, which led to:

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \mathbf{J}(\mathbf{x}')e^{-ikr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'}d\mathbf{x}'$$

- We further expanded the source into multipoles and found that electric dipole radiation in the far field is:

$$\mathbf{A}(\mathbf{x},t)=-\frac{i\omega\mu_0}{4\pi}\mathbf{p}\frac{e^{i(kr-\omega t)}}{r} \quad \text{where} \quad \mathbf{p}=\int \mathbf{x}'(\rho(\mathbf{x}'))d\mathbf{x}'$$

which leads to the fields:

$$\mathbf{B}=\frac{\mu_0 c k^2 p}{4\pi}(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{i(kr-\omega t)}}{r}$$

$$\mathbf{E}=-\frac{\mu_0 c^2 k^2 p}{4\pi}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{i(kr-\omega t)}}{r}$$

- The unit vector in the propagation direction $\hat{\mathbf{k}}$ shown above should not be confused with the unit vector in the *z* direction.
- The dipole moment $\mathbf{p}$ in these equations is the peak instantaneous electric dipole moment since the harmonic time dependence has already been written out separately.
- These are transverse spherical waves traveling radially outwards.
- It should be noted that for radiating dipoles, the electric field oscillates in a plane which is always parallel to the dipole, while the magnetic field oscillates normal to the plane containing the dipole. This means that the waves are linearly polarized. This has important consequences in areas such as radio broadcast and radar imaging.
- A straight antenna that is oriented vertically will create vertically-polarized radiation.
- To create circularly-polarized waves, we must either use spiral antennas or two perpendicular, straight antennas that are 90 degrees out of phase.
- Similarly, in lasers where the transitioning electrons behave approximately like a dipole, and the transitions are all aligned in space, the laser radiation is linearly polarized.
- The find the radiated power, we turn to the Poynting vector:

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}$$

- In a previous lecture, we found that if we apply this to harmonic waves in free space and take the time-average, we have:

$$<\mathbf{S}>=\frac{1}{2\mu_0}\Re[\mathbf{E}\times\mathbf{B}^*]$$

- For waves created by a localized source, the only energy actually radiated away is carried by the waves traveling radially outwards. We therefore dot the time-averaged Poynting vector with the radial unit vector to isolate just the radiated energy:

$$\hat{\mathbf{k}}\cdot<\mathbf{S}>=\frac{1}{2\mu_0}\Re[\hat{\mathbf{k}}\cdot(\mathbf{E}\times\mathbf{B}^*)]$$

- For waves radiated by localized sources, this expression in the far field will always have a trivial radial dependence of $1/r^2$. Therefore, we multiply both sides by r^2 to normalize away the radial dependence:

$$r^2\hat{\mathbf{k}}\cdot<\mathbf{S}>=\frac{1}{2\mu_0}r^2\Re[\hat{\mathbf{k}}\cdot(\mathbf{E}\times\mathbf{B}^*)]$$

- We give this radially-normalized, time-averaged radiated energy flux a special name: the radiated power per unit solid angle $\frac{dP}{d\Omega}$.

$$\boxed{\frac{dP}{d\Omega}=\frac{1}{2\mu_0}r^2\Re[\hat{\mathbf{k}}\cdot(\mathbf{E}\times\mathbf{B}^*)]}$$ *Radiated Power per Unit Solid Angle*

- The equation above is somewhat general and applies to all waves radiated into the far field by a localized source.
- We now apply this equation to electric dipole radiation by inserting our expressions for the electric and magnetic field:

$$\frac{dP}{d\Omega}=\frac{1}{2\mu_0}r^2\Re\left[\hat{\mathbf{k}}\cdot\left(\left[-\frac{\mu_0c^2k^2p}{4\pi}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{i(kr-\omega t)}}{r}\right]\times\left[\frac{\mu_0ck^2p}{4\pi}(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{-i(kr-\omega t)}}{r}\right]\right)\right]$$

$$\frac{dP}{d\Omega}=\frac{c}{2\mu_0}\frac{\mu_0^2c^2k^4p^2}{16\pi^2}\Re[\hat{\mathbf{k}}\cdot([-\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{p}})]\times[(\hat{\mathbf{k}}\times\hat{\mathbf{p}})])]$$

$$\boxed{\frac{dP}{d\Omega}=\frac{ck^4p^2}{32\pi^2\epsilon_0}\sin^2\theta}$$ where θ is the angle between **p** and **k**.

- We see that most of the electromagnetic energy is radiated sideways, perpendicular to the electric dipole moment **p**, and none is radiated along the dipole's axis.
- To find the total power radiated in all directions, we simply integrate over all angles:

$$P = \int_S \left[\frac{dP}{d\Omega} \right] d\Omega$$

$$P = \int_0^{2\pi} \int_0^{\pi} \left[\frac{dP}{d\Omega} \right] \sin\theta \, d\theta \, d\phi$$

$$P = \int_0^{2\pi} \int_0^{\pi} \left[\frac{c k^4 p^2}{32 \pi^2 \epsilon_0} \sin^2\theta \right] \sin\theta \, d\theta \, d\phi$$

$$P = \frac{c k^4 p^2}{16 \pi \epsilon_0} \int_0^{\pi} \sin^3\theta \, d\theta$$

$$\boxed{P = \frac{c k^4 p^2}{12 \pi \epsilon_0}}$$ *Total Power Radiated by a Harmonically Oscillating Electric Dipole*

- To find the total energy radiated in a given time span *T*, we integrate the power over the time span:

$$E = \int_0^T P \, dt = PT \quad \text{if } P \text{ is constant in time}$$

$$\boxed{E = \frac{c k^4 p^2}{12 \pi \epsilon_0} T}$$

2. Small Center-Fed Linear Antennas

- Consider a linear antenna of length *d*, placed along the *z* axis and centered on the origin. The current is fed in at the center so that the current profile on the top half matches that of the bottom half.
- The current must be zero at the ends of the antenna, and a maximum at the center. Since the antenna is small compared to the wavelength of the waves it is emitting, the current distribution can be taken to be linear in space. It should therefore have the form:

$$I(z,t) = I_0 \left(1 - \frac{2|z|}{d} \right) e^{-i\omega t}$$

- Using the continuity equation, we can find the charge density:

$$\frac{\partial \rho(\mathbf{x},t)}{\partial t} = -\nabla \cdot \mathbf{J}(\mathbf{x},t)$$

- Assume harmonic time dependence: $\mathbf{J}(\mathbf{x},t) = \mathbf{J}(\mathbf{x}) e^{-i\omega t}$, $\rho(\mathbf{x},t) = \rho(\mathbf{x}) e^{-i\omega t}$ so that the continuity equation becomes:

$$i \omega \rho(\mathbf{x}) = \nabla \cdot \mathbf{J}(\mathbf{x})$$

- The current density in this case only has a vector component in the z direction, leading to:

$$i\omega\rho(\mathbf{x}) = \frac{dJ(\mathbf{x})}{dz}$$

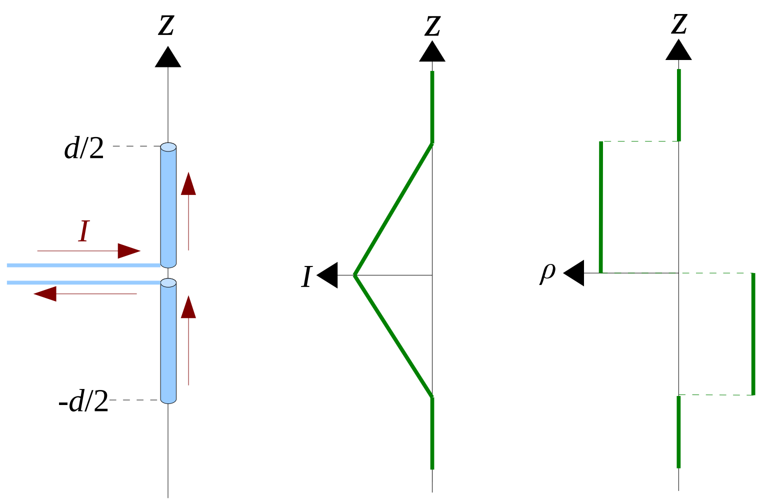
- If we assume the wire to be infinitesimally thin, we have $J(\mathbf{x}) = \delta(x)\delta(y)I(z)$ so that the continuity equation becomes:

$$\rho(\mathbf{x}) = -\frac{i}{\omega}\frac{dI(z)}{dz}\delta(x)\delta(y)$$

- Now plug into the above equation our expression for the current distribution to find:

$$\rho(\mathbf{x}) = sgn(z)\frac{2iI_0}{\omega d}\delta(x)\delta(y) \quad \text{where } sgn(z) = +1 \text{ for } z > 0 \text{ and } sgn(z) = -1 \text{ for } z < 0$$

- We see that the charge distribution is constant across one half of the antenna, but that the two halves of the antenna always have opposite charge.



- If the wavelength associated with the harmonic oscillation of the current is much larger than the length d of the antenna, we are only interested in fields at distances much greater than both, $d << \lambda << r$, and we can make a physical argument that the dipole moment will dominate. Therefore, we can use all the dipole far-field results found previously.
- The peak instantaneous dipole moment of this antenna is:

$$\mathbf{p} = \int \mathbf{x}'(\rho(\mathbf{x}'))d\mathbf{x}'$$

$$\mathbf{p} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2}(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}})\rho(x', y', z')dz'dx'dy'$$

$$\mathbf{p} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-d/2}^{d/2}(x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} + z'\hat{\mathbf{z}})sgn(z')\frac{2iI_0}{\omega d}\delta(x')\delta(y')dz'dx'dy'$$

$$\mathbf{p} = \hat{\mathbf{z}} \frac{2 i I_0}{\omega d} \int_{-d/2}^{d/2} z' sgn(z') dz'$$

$$\mathbf{p} = \hat{\mathbf{z}} \frac{2 i I_0}{\omega d} \left[- \int_{-d/2}^{0} z' dz' + \int_{0}^{d/2} z' dz' \right]$$

$$\mathbf{p} = \hat{\mathbf{z}} \frac{2 i I_0}{\omega d} \left[\frac{d^2}{4} \right]$$

$$\boxed{\mathbf{p} = \hat{\mathbf{z}} \frac{i I_0 d}{2 \omega}}$$

- Plugging this in to our equations gives:

$$\boxed{\mathbf{B} = -\frac{\mu_0 k d I_0}{8\pi} \sin\theta \frac{e^{i(kr - \omega t + \pi/2)}}{r} \hat{\boldsymbol{\phi}}}$$

$$\boxed{\mathbf{E} = -\frac{\mu_0 c k d I_0}{8\pi} \sin\theta \frac{e^{i(kr - \omega t + \pi/2)}}{r} \hat{\boldsymbol{\theta}}}$$

$$\boxed{\frac{dP}{d\Omega} = \frac{I_0^2 k^2 d^2}{128 \pi^2 \epsilon_0 c} \sin^2\theta}$$

$$\boxed{P = \frac{I_0^2 k^2 d^2}{48 \pi \epsilon_0 c}}$$