Lecture 6 Notes, Electromagnetic Theory II

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1. Radiation Introduction

- We have learned about the propagation of waves, now let us investigate how they are generated.
- Physically speaking, whenever the electric field changes in time, it creates a magnetic field which changes in time, which creates an electric field which changes in time, etc.
- This creates a chain reaction that then propagates away from the system and exists independent of it.
- Every time the fields change non-linearly in time, waves are radiated.
- Any acceleration of an electric charge creates a non-linearly changing electric field. Therefore, accelerating charges radiate electromagnetic waves.
- The only system that does not radiate is a system where all the charges are completely at rest with respect to some inertial frame.
- In the real world, no charges are ever completely at rest. Every object has some temperature, even if very small, which means its atoms and electrons are constantly experiencing thermal motion. These moving charges radiate waves.
- Therefore, all materials are constantly radiating electromagnetic waves because of their thermal motion.
- The thermal radiation emitted by an object tends to be random and is often viewed as undesired noise that must be overcome by a meaningful signal.
- "Electrostatics" is an approximation where the radiation due to thermal motion and other motion of charges is small enough to be considered negligible.
- Unfortunately, the types of radiation and their sources are quite numerous and involved and we only have time in this course to cover the simplest forms of radiation.

2. General Solution for Radiation

- For the purpose of understanding radiation, assume we have charges oscillating in free space. There are no boundaries or materials present.
- The Maxwell-Ampere Law states that currents and changing electric fields give rise to curling magnetic fields:

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

- Define potentials in the usual way, $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{E}=-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}$, and plug them in:

$$\nabla\times(\nabla\times\mathbf{A})=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial}{\partial t}\left(-\nabla\Phi-\frac{\partial\mathbf{A}}{\partial t}\right)$$

$$\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}=\mu_0\mathbf{J}-\frac{1}{c^2}\nabla\frac{\partial}{d t}\Phi-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}$$

- Because the fields are defined in terms of the derivatives of the potentials, there is some freedom in choosing potentials.
- Any specific choice of potentials is called a “gauge”.
- The Maxwell equations are “gauge invariant”, meaning that they give the same answer in the end even if we switch from one gauge to the next.
- Let us use the Lorenz gauge, defined by:

$$\nabla\cdot\mathbf{A}=-\frac{1}{c^2}\frac{\partial\Phi}{\partial t}$$

- This causes two terms in the Maxwell-Ampere Law to cancel, leaving us with:

$$\boxed{\nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu_0\mathbf{J}}$$

Maxwell-Ampere Law in Free Space in the Lorenz Gauge

- In the Lorenz gauge, the Maxwell-Ampere Law thus reduces to a wave equation.
- This is an inhomogeneous partial differential equation. Its general solution is the sum of the solution to the corresponding homogeneous equation and the solution to the inhomogeneous equation:

$$\mathbf{A}=\mathbf{A}_{\text{homo}}+\mathbf{A}_{\text{inhomo}}$$

- The homogeneous equation is just the one where there are no sources, **J** = 0. We have already solved that equation. Its particular solution is transverse plane waves propagating in space.
- The inhomogeneous equation corresponds to sources creating fields.
- The general solution then is just waves that were already propagating along plus new waves and fields that are created by electric charges.
- Let us find the Green function solution to the inhomogeneous Maxwell-Ampere Law.
- In the usual way, if there are no boundaries present, the Green function solution is:

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int\int G(\mathbf{x},t,\mathbf{x}',t')\mathbf{J}(\mathbf{x}',t')\,d\mathbf{x}'\,dt'$$

where the Green function satisfies:

$$\nabla^2 G-\frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}=-4\pi\,\delta(\mathbf{x}-\mathbf{x}')\,\delta(t-t')$$

- Using Fourier Transforms and after much work, the solution is found to be (the full derivation was done last semester in Lecture 14).

$$G=\frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t\pm\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)$$

- This is essentially the static Green's function times a Dirac delta which ensures causality.
- Causality means that it takes time for the effects to propagate. The effect of some source's

action at a location **x**' and time *t*' cannot be felt at the observation point **x** until a later time $t = t' + |\mathbf{x} - \mathbf{x}'|/c$.

- The plus or minus accounts for an incoming or an outgoing signal.
- Now use this solution for *G* to get the final solution for the potential:

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\delta\left(t'-t\pm\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)\mathbf{J}(\mathbf{x}',t')\,d\mathbf{x}'\,dt'$$

$$\boxed{\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}\left(\mathbf{x}',t\mp\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)d\mathbf{x}'}$$

- This is essentially the magnetostatics solution but with the time shifted to ensure causality.
- For the purpose of studying radiation, we only care about the outgoing signal traveling towards us (the retarded signal):

$$\boxed{\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}\left(\mathbf{x}',t-\frac{1}{c}|\mathbf{x}-\mathbf{x}'|\right)d\mathbf{x}'}$$

3. Radiation of Harmonically Oscillating Sources

- Most radiation systems of interest involve a localized oscillating source. We can always build-up a non-localized source as a sum of localized sources, so the analysis below still has general applicability.
- A localized source is a system of electric charges and currents that all reside within some finite sphere of radius *d* such that our observation distance *r* from the source is external to the source: $r >> d$.
- As has been done previously, the simplest way to handle the time dependence of the radiation is to view the final solution as a superposition of single-frequency waves.
- Therefore, we only need to solve the problem for a single frequency and then in the end we can sum them in the appropriate way to get any arbitrary solution.
- The oscillating sources can be broken down into frequency components:

$$\mathbf{J}(\mathbf{x},t)=\mathbf{J}(\mathbf{x})e^{-i\omega t}$$

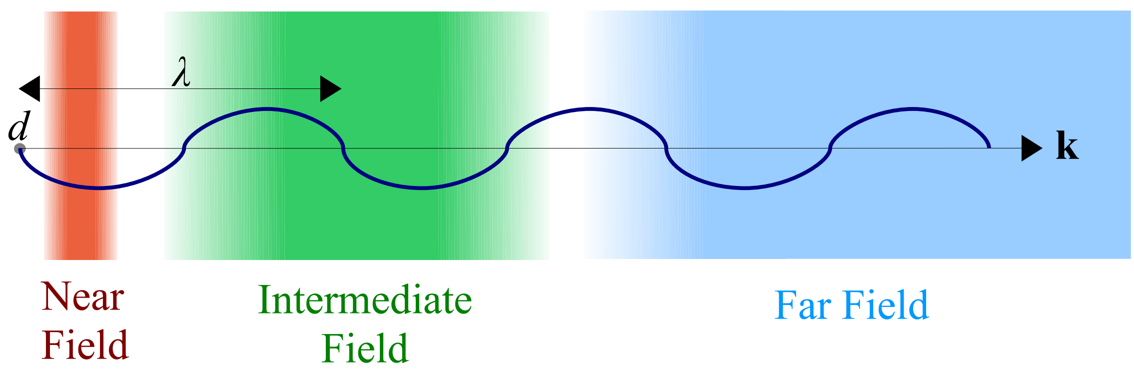
- Plugging in this current density into the general solution above, we find:

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\int\frac{1}{|\mathbf{x}-\mathbf{x}'|}\mathbf{J}(\mathbf{x}')e^{-i\omega(t-|\mathbf{x}-\mathbf{x}'|/c)}d\mathbf{x}'$$

$$\boxed{\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}e^{-i\omega t}\int\mathbf{J}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'}$$

- This result is general for any harmonically-varying source.
- Note that at this point, **J** represents the peak current distribution, i.e. the current distribution when its oscillation in time hits its peak. We have already explicitly extracted the harmonic time-dependence of **J** and brought it out front.

- To investigate the behavior of a localized harmonically oscillating source, we will expand this equation into a series and use several limiting procedures to only keep a few terms.
- We will do this in different regions of space defined by their proximity to the source.
- These regions only have meaning if the associated wavelength of the oscillation of the charges, $\lambda = 2\pi c/\omega$, is much larger than the dimensions of the source, $\lambda >> d$. This is called the long-wavelength approximation. If we were to not make this approximation, we would have to take into account the fact that the waves from one side of the source can travel over to the other side of the source and scatter off of it.



- Define the near-field region as the region of space that is very far from the source but still much smaller than a wavelength of the light: $d << r << \lambda$ which leads to $kr << 1$.
- Define the intermediate-field region as the region where the observation distance is on the order of the wavelength of the light: $r \approx \lambda$.
- Define the far-field region as the region where the observation distance is much greater than the wavelength of the light: $r >> \lambda$ which leads to $kr >> 1$.

4. Near-Field Radiation

- In the near field, the exponential becomes one and the solution becomes:

$$\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int \mathbf{J}(\mathbf{x}') \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}'$$

$$\boxed{\mathbf{A}(\mathbf{x},t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int \mathbf{J}(\mathbf{x}') \frac{1}{|\mathbf{x}-\mathbf{x}'|} d\mathbf{x}'}$$ *Near Field Solution*

- The spatial part is the exact same solution as found in magnetostatics.
- In the near field, therefore, the fields respond instantaneously to the source. The problem is essentially the static solution, varying harmonically in time.
- Because the methods for solving magnetostatics problems have already been covered, there is not much left to be said here.
- In the near-field, electrostatic (capacitive) and magnetostatic effects dominate.

5. Intermediate-Field Radiation

- In this region, we can not make any approximations.
- The exponential and inverse distance factors are expanded in vector spherical harmonics and integrated term by term.

- The methods of vector spherical harmonic expansion are complex and involved. There is insufficient time in this course to cover this subject to any satisfaction.
- In the intermediate-field, the inductive effects dominate.
- In the far-field, radiation effects dominate.

6. Far-Field Radiation

- The far field is defined as $kr >> 1$ (or written equivalently as $r >> \lambda$).
- The long-wavelength approximation tells us that $\lambda >> d$. Since all points r' in the source are contained within the sphere of diameter d, this also means that $\lambda >> r'$.
- Combining the long-wavelength approximation and the far-field approximation, we therefere see that $r >> r'$ (or equivalently $1 >> r'/r$).
- Thus any time we have a sum of different powers of r'/r, we can drop the higher powers of r'/r since they will be vanishingly small.
- Let us use the notation $r = |\mathbf{x}|$ and $r' = |\mathbf{x}'|$.
- Using the law of cosines, we can expand the separation distance:

$$|\mathbf{x}-\mathbf{x}'|=\sqrt{r^2+r'^2-2rr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'}$$

- We now try to get everything in terms of powers of r'/r so that we can drop the higher powers.

$$|\mathbf{x}-\mathbf{x}'|=r\sqrt{1+\left(\frac{r'}{r}\right)^2-2\left(\frac{r'}{r}\right)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'}$$

- Expand this equation into a binomial series using $\sqrt{1+u}=1+\frac{1}{2}u-\frac{1}{8}u^2+...$:

$$|\mathbf{x}-\mathbf{x}'|=r\left(1+\frac{1}{2}\left[\left(\frac{r'}{r}\right)^2-2\left(\frac{r'}{r}\right)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right]-\frac{1}{8}\left[\left(\frac{r'}{r}\right)^2-2\left(\frac{r'}{r}\right)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right]^2+...\right)$$

- Because of $r'/r << 1$, we can drop all terms except the first two. (If we kept only the first term we would end up with dipole radiation. However, right now we are trying to get a general expression in the far-field that is valid for all multipole moments.)

$$|\mathbf{x}-\mathbf{x}'|=r\left(1-\left(\frac{r'}{r}\right)\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'\right)$$

$$|\mathbf{x}-\mathbf{x}'|=r-r'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'$$

- Using this approximation, the solution for the vector potential in the far field becomes:

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}e^{-i\omega t}\int \mathbf{J}(\mathbf{x}')\frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}d\mathbf{x}'$$

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}e^{-i\omega t}\int \mathbf{J}(\mathbf{x}')\frac{e^{ik(r-r'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')}}{(r-r'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')}d\mathbf{x}'$$

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}e^{i(kr-\omega t)}\int \mathbf{J}(\mathbf{x}')\frac{e^{-ikr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'}}{(r-r'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')}d\mathbf{x}'$$

- Because of $r'/r \ll 1$, we can drop the second term in the denominator, leading to:

$$\boxed{\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \mathbf{J}(\mathbf{x}')e^{-ikr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'}d\mathbf{x}'}$$ *Far Field Solution*

- The factor outside of the integral tells us that we have traveling waves propagating spherically outwards in the far field.
- Note that the integral does not depend on the observation distance *r*. At the same time, the integral is the only part of the equation that does depend on the polar angle and azimuthal angle. Therefore, the integral specifies the angular radiation pattern (or "antenna pattern"). Once a wave is in the far-field, its angular pattern stays the same as the waves travels outwards.
- Also note that that current density **J** in the integral is just the spatial part of the current density (i.e. the peak current density) because its time dependence has already been extracted and written out front.

7. Electric Dipole Radiation in the Far Field

- Let us expand the exponential in the integral according to $e^x=1+x+\frac{1}{2}x^2+...$

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \mathbf{J}(\mathbf{x}')e^{-ikr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}'}d\mathbf{x}'$$

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \mathbf{J}(\mathbf{x}')(1+(-ikr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')+\frac{1}{2}(-ikr'\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}')^2+...)d\mathbf{x}'$$

- Each power of *kr*' in the expansion corresponds to one of the multipole moments of the source.
- According to the long-wavelength approximation $kr' \ll 1$ so that higher powers of *kr*' are vanishingly small. That means that we only need to keep the first few multipole moments and we will have a good solution.
- The first term, or magnetic monopole term is:

$$\mathbf{A}(\mathbf{x},t)=\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \mathbf{J}(\mathbf{x}')d\mathbf{x}'$$

- In magnetostatics, the magnetic monopole integral is zero because the currents do not diverge.
- In electrodynamics, the currents can diverge, so there can be a non-zero monopole term.
- Perform an integration by parts on the integral to find:

$$\mathbf{A}(\mathbf{x},t)=-\frac{\mu_0}{4\pi}\frac{e^{i(kr-\omega t)}}{r}\int \mathbf{x}'(\nabla'\cdot\mathbf{J}(\mathbf{x}'))d\mathbf{x}'$$

- In magnetostatics, we had $\nabla'\cdot\mathbf{J}(\mathbf{x}')=0$, which made the magnetic monopole go away.
- However, in electrodynamics, we can have a diverging current according to the continuity equation:

$$\nabla' \cdot \mathbf{J}(\mathbf{x}', t) = -\frac{\partial \rho(\mathbf{x}', t)}{\partial t}$$

- Using the continuity equation and applying harmonic dependence on time we have:

$$\nabla' \cdot \mathbf{J}(\mathbf{x}') e^{-i\omega t} = -\frac{\partial}{\partial t}(\rho(\mathbf{x}') e^{-i\omega t})$$

$$\nabla' \cdot \mathbf{J}(\mathbf{x}') = i\omega \rho(\mathbf{x}')$$

- Plugging this into the magnetic monopole equation transforms it from magnetic monopole radiation to electric dipole radiation. Of course, they are really the same thing in electrodynamics because the electric and magnetic fields are coupled.

$$\mathbf{A}(\mathbf{x}, t) = -\frac{i\omega\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \int \mathbf{x}'(\rho(\mathbf{x}')) d\mathbf{x}'$$

$$\boxed{\mathbf{A}(\mathbf{x}, t) = -\frac{i\omega\mu_0}{4\pi} \mathbf{p} \frac{e^{i(kr-\omega t)}}{r}} \quad \text{where} \quad \boxed{\mathbf{p} = \int \mathbf{x}'(\rho(\mathbf{x}')) d\mathbf{x}'}$$

Electric Dipole Radiation

- The property **p** is the electric dipole moment as already encountered in electrostatics.
- Note that the electric dipole moment **p** and charge density ρ shown in the equations above are the peak, instantaneous values since the harmonic time-dependence of both has already been explicitly factored out and written out separately. To make this point more obvious, we could rewrite the solution in the form:

$$\mathbf{A}(\mathbf{x}, t) = -\frac{i\omega\mu_0}{4\pi} \mathbf{p}(t) \frac{e^{ikr}}{r} \quad \text{where} \quad \mathbf{p}(t) = \int \mathbf{x}'(\rho(\mathbf{x}', t)) d\mathbf{x}' \quad \text{and} \quad \rho(\mathbf{x}', t) = \rho(\mathbf{x}') e^{-i\omega t}$$

- Calculating the fields from this vector potential:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{B} = \nabla \times \left(-\frac{i\omega\mu_0}{4\pi} \mathbf{p} \frac{e^{i(kr-\omega t)}}{r} \right)$$

$$\mathbf{B} = -\frac{i\omega\mu_0}{4\pi} \nabla \times \left(\mathbf{p} \frac{e^{i(kr-\omega t)}}{r} \right)$$

- We now use the identity shown below, which can be proved by expanding everything into components and remembering that by r we mean $r = |\mathbf{x}|$:

$$\nabla \times [\mathbf{p} f(r)] = (\hat{\mathbf{x}} \times \mathbf{p}) \frac{\partial f(r)}{\partial r}$$

$$\mathbf{B} = -\frac{i\omega\mu_0}{4\pi} (\hat{\mathbf{x}} \times \mathbf{p}) \frac{\partial}{\partial r} \left(\frac{e^{i(kr-\omega t)}}{r} \right)$$

$$\mathbf{B} = -\frac{i\omega\mu_0}{4\pi}(\hat{\mathbf{x}}\times\mathbf{p})\left(\frac{-e^{i(kr-\omega t)}}{r^2} + ik\frac{e^{i(kr-\omega t)}}{r}\right)$$

- In the far-field, only the last term contributes since *r* is so large, leading to:

$$\mathbf{B} = \frac{\mu_0 c k^2}{4\pi}(\hat{\mathbf{x}}\times\mathbf{p})\frac{e^{i(kr-\omega t)}}{r}$$

- It is important to note that the source is at the origin and the waves propagate radially outward, so that the radial vector and the propagation vector point in the same direction: $\hat{\mathbf{x}} = \hat{\mathbf{k}}$.

$$\boxed{\mathbf{B} = \frac{\mu_0 c k^2 p}{4\pi}(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{i(kr-\omega t)}}{r}}$$ *Magnetic Field Due to Electric Dipole Radiation*

- The unit vector pointing in the propagation direction used here must not be confused with the unit vector in the *z* direction, although they may be labeled the same.
- Again note that the electric dipole moment **p** shown in the equation above is the peak instantaneous dipole moment, since its harmonic time dependence has already been written out separately.
- This equation has three main pieces.
- The first factor (including all variables up to the first parenthesis) gives the overall strength of the magnetic field radiated by an electric dipole oscillating harmonically. As we would expect, increasing the strength of the dipole **p** increases the magnetic field strength. Perhaps unexpected though is that increasing the wavenumber of the oscillation increases the field strength.
- The second factor in the equation (the term in parentheses) completely determines the direction of the magnetic field. It is perpendicular to the direction of propagation, as expected of a transverse traveling wave. It is also perpendicular to the electric dipole moment vector. This is expected because the magnetic field should perpendicular to the electric field, which should be generally aligned with the electric dipole moment.
- The second factor also modulates the field according to $|\hat{\mathbf{k}}\times\hat{\mathbf{p}}| = \sin\gamma$ were γ is the angle between the two vectors. For instance, the propagation direction that is along the axis of the dipole has a 0° angle and therefore there are no waves propagating in this direction.
- The last factor in the equation modulates the overall strength such that it is an oscillating traveling wave that travels radially outward and dies in strength as it travels and spreads out.
- Let us now find the corresponding electric field.
- The Maxwell-Ampere Law in free space states:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

- In the far field we are external to the localized current distribution so that **J** = 0, leading to:

$$\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}$$

- Apply harmonic time dependence and solve for the electric field:

$$\mathbf{E}=i\frac{c}{k}\nabla\times\mathbf{B}$$

- If we take this general result and apply it to the magnetic field of the electric dipole radiation, we get:

$$\mathbf{E}=i\frac{c}{k}\nabla\times\left(\frac{\mu_0 c k^2 p}{4\pi}(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{i(kr-\omega t)}}{r}\right)$$

$$\mathbf{E}=i\frac{\mu_0 c^2 k p}{4\pi}\nabla\times\left((\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{i(kr-\omega t)}}{r}\right)$$

$$\mathbf{E}=i\frac{\mu_0 c^2 k p}{4\pi}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{\partial}{\partial r}\left(\frac{e^{i(kr-\omega t)}}{r}\right)$$

- Evaluating the derivative and dropping the higher order term, we find:

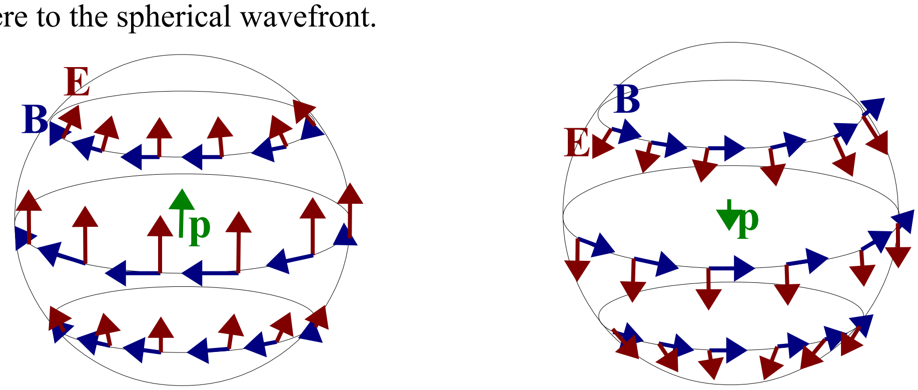
$$\mathbf{E}=-\frac{k^2 p}{4\pi\epsilon_0}\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\hat{\mathbf{p}})\frac{e^{i(kr-\omega t)}}{r}$$

Electric Field Due to Electric Dipole Radiation

- Again, the unit vector pointing in the propagation direction must not be confused with the unit vector in the *z* direction, although they are labeled the same.
- As expected, the electric field points perpendicular to the magnetic field, perpendicular to the wavevector, and is in the same plane as the electric dipole moment vector. We therefore have standard transverse traveling waves.
- This equation may be more understandable if we align the dipole with the *z*-axis, use spherical coordinates, and look at only the magnitude of the electric field vector:

$$E=\frac{\mu_0 c^2 k^2 p}{4\pi}\sin\theta\frac{e^{i(kr-\omega t)}}{r}$$

- This equation tells as that waves are not radiated along the axis of electric dipoles and are most strongly radiated perpendicular to the dipole.
- We can sketch these results for better understanding. For ease of illustration, imagine the dipole oriented along the *z* axis and imagine we can take snapshots at some fixed radius at some time and then at a later time. The electric field and magnetic field vectors lie tangential everywhere to the spherical wavefront.



APPENDIX

Let us look at the mathematical derivation of simple traveling waves in different coordinate systems. The wave equation in free space when no sources are present is:

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

A "wavefront" is a surface in space along which the electric field has the same phase at a given moment in time. In other words, a wavefront is a surface that connects all the adjacent points in space that are reaching their peak electric field strength at the same time. Rather than restrict ourselves to the specific cases of waves that have constant phase *and* amplitude across their wavefronts, we will look at the more general cases of waves that have only constant phase across their wavefronts. With this in mind, we now use the phrase "plane wave" to indicate a wave with planar wavefronts and not necessarily a wave with constant amplitude across its wavefront. In addition to the wave equation, the complete set of Maxwell's equations in their standard form restricts what types of field patterns can have wavefronts with a simple shape. However, we will not investigate here all the restrictions placed on field patterns by Maxwell's equation. Instead, we seek here to only get a rough picture of what electromagnetic waves looks like with wavefronts of various shapes.

Plane Waves

Let us first look at electromagnetic waves with planar wavefronts. Expand the wave equation into rectangular coordinates and focus on the i^{th} vector component of the electric field:

$$\frac{\partial^2 E_i}{\partial x^2} + \frac{\partial^2 E_i}{\partial y^2} + \frac{\partial^2 E_i}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} = 0$$

As should be obvious, there is complete symmetry between the various rectangular coordinates. Therefore, if we align the planar wavefronts with the *x*-*y* plane, our results are still general. Thus, for definiteness, we choose the wavefronts to be parallel to the *x*-*y* plane and traveling in the positive or negative *z* direction. To keep the discussion general, we relabel the first two terms in the above wave equation as the transverse Laplacian of the electric field:

$$\nabla_t^2 E_i + \frac{\partial^2 E_i}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} = 0 \qquad \text{where } \nabla_t^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The word "transverse" used here means that that Laplacian only acts on the dimensions along the extent of a wavefront, i.e. the dimensions that are transverse to the direction the waves are traveling. To ensure constant phase across each wavefront, the part of the electric field solution that depends on x and y (which we label $E_{i,t}$) must be real-valued. Since the wave is traveling in the z direction, we expect the electric field's dependence on z and t to be allowed to be harmonic. Putting these concepts together, we get our trial solution:

$$E_i = E_{i,t}(x, y)\, e^{\pm i k z - i \omega t} \quad \text{where } E_{i,t} \text{ is a real-valued function depending only on transverse dimensions}$$

Inserting this trial solution into the wave equation, we find:

$$\nabla_t^2 E_{i,t} = (k^2 - \omega^2/c^2) E_{i,t}$$

Combine the two parameters on the right-hand side of the above equation into one parameter ($-\kappa^2$), leading to:

$$\nabla_t^2 E_{i,t} + \kappa^2 E_{i,t} = 0$$

This equation just specifies the transverse behavior of the electric field and this is as far as we want to go. In summary, the particular solution to the wave equation for planar wavefronts is:

$\boxed{E_i = E_{i,t}(x, y) e^{\pm ikz - i\omega t}}$ where $k = \sqrt{\omega^2/c^2 - \kappa^2}$ and $E_{i,t}$ is real-valued and obeys $\nabla_t^2 E_{i,t} = -\kappa^2 E_{i,t}$

We call this a "plane" wave because for a given z and t, the electric field has the same phase across the entire flat plane in the x and y directions. Conceptually, a plane wave could by created by a sheet charge distribution across an infinite plane that is oscillating.

Cylindrical Waves

Let us next look at electromagnetic waves with cylindrical wavefronts that are traveling towards or away from the z axis. We expand the wave equation into cylinder coordinates and relabel the transverse terms in terms of the transverse Laplacian:

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial E_i}{\partial \rho}\right) + \nabla_t^2 E_i - \frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2} = 0 \quad \text{where} \quad \nabla_t^2 = \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2}$$

Since we are taking the wavefronts to be cylindrical, the part of the electric field that depends on the transverse dimensions (the azimuthal angle and z) must be real-valued. Since the wave is traveling in the cylindrical radial direction, we expect the electric field's dependence on t to be allowed to be harmonic, and the its dependence on ρ to be complex-valued. These concepts suggest a trial solution of the form:

$$E_i = R(\rho) E_{i,t}(\phi, z) e^{-i\omega t} \quad \text{where } E_{i,t} \text{ is real-valued and } R \text{ is complex-valued}$$

Plugging this form into the wave equation and using separation of variables, we get:

$$\frac{1}{R(\rho)}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R(\rho)}{\partial \rho}\right) - \frac{m^2}{\rho^2} - k_z^2 + \frac{\omega^2}{c^2} = 0 \quad \text{where} \quad -\left(\frac{m^2}{\rho^2} + k_z^2\right) E_{i,t} = \nabla_t^2 E_{i,t}$$

The equation on the right just specifies the transverse behavior of the electric field and it is as far as we want to go with the transverse behavior. For the equation on the left, relabel the last two terms as the squared radial wavevector k^2 according to $k^2 = -k_z^2 + \omega^2/c^2$ so that the equation becomes:

$$\frac{1}{R(\rho)}\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R(\rho)}{\partial \rho}\right) - m^2 \frac{1}{\rho^2} + k^2 = 0$$

Make the substitution $u = k\rho$ and rearrange to find:

$$u^2 \frac{\partial^2 R(u)}{\partial u^2} + u \frac{\partial R(u)}{\partial u} + (u^2 - m^2) R(u) = 0$$

This equation is Bessel's differential equation and the solutions are Bessel functions:

$R(u) = a J_m(u) + b N_m(u)$ which leads to:

$$R(\rho) = a J_m(k\rho) + b N_m(k\rho)$$

In order to get traveling waves in the radial direction, R must be complex-valued. However, since the Bessel functions are real-valued, the forms above for R are not complex-valued. We can transform them to Hankel functions which are complex valued. We can do this because of the sum of any solutions to a linear differential equation is also a solution of the differential equation. The solution now becomes:

$R(\rho) = a H_m^{(1)}(k\rho) + b H_m^{(2)}(k\rho)$ where $H_m^{(1)} = J_m + i Y_m$, $H_m^{(2)} = J_m - i Y_m$

For large ρ (far away from the z axis), the Hankel functions asymptotically approach the forms:

$H_m^{(1)}(x) = C \frac{1}{\sqrt{x}} e^{-ix}$ and $H_m^{(2)}(x) = D \frac{1}{\sqrt{x}} e^{ix}$

Therefore, for large ρ, the radial part of the electric field solution becomes:

$$R(\rho) = a \frac{1}{\sqrt{\rho}} e^{\pm i k \rho}$$

In summary, the particular solution to the wave equation for cylindrical wavefronts far from the z axis is:

$\boxed{E_i = E_{i,t}(\phi, z) \frac{e^{\pm i k \rho - i \omega t}}{\sqrt{\rho}}}$ where $k = \sqrt{\omega^2 / c^2 - k_z^2}$ and $E_{i,t}$ is real and obeys $\nabla_t^2 E_{i,t} = -\left(\frac{m^2}{\rho^2} + k_z^2\right) E_{i,t}$

These equations tell us that the particular solution to the wave equation with cylindrical wavefronts far away from the z axis is sinusoidal traveling waves that are traveling outwards or inwards in the radial direction. We call these waves “cylindrical waves” since on a given cylindrical surface centered on the z axis, the electric field has constant phase across the surface. Conceptually, cylindrical waves are created by an infinitely-long, straight, oscillating line charge. The factor of $1/\sqrt{\rho}$ is needed in the above equation to ensure conservation of energy. Since the total energy carried along by an expanding wavefront must be constant, and since the total energy is the energy density times the area, the energy density must decrease at the rate that the cylinder's area is increasing. Since a cylinder's area is $2\pi\rho h$, and thus increases linearly with radius, the energy density must be proportional to $1/\rho$. Because the electric field is proportional to the square root of the energy density, the electric field strength must be proportional to $1/\sqrt{\rho}$. Thus we see that energy conservation is the explanation of the appearance of this factor in the solution above.

Spherical Waves

Let us next look at electromagnetic waves with spherical wavefronts. If we expand the wave equation into spherical coordinates and relabel the parts that operate on the transverse dimensions as the transverse Laplacian, we get:

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r E_i)+\nabla_t^2 E_i-\frac{1}{c^2}\frac{\partial^2 E_i}{\partial t^2}=0 \quad \text{where} \quad \nabla_t^2=\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

Since we are taking the wavefronts to be spherical, the part of the electric field that depends on the transverse dimensions (the azimuthal angle and polar angle) must be real-valued. Since the wave is traveling in the spherical radial direction, we expect the electric field's dependence on *t* to be allowed to be harmonic, and its dependence on *r* to be complex-valued. These concepts suggest a trial solution of the form:

$$E_i=R(r)E_{i,t}(\phi,\theta)e^{-i\omega t} \quad \text{where } E_{i,t} \text{ is real-valued and } R \text{ is complex-valued}$$

Plugging this form into the wave equation and using separation of variables, we get:

$$\frac{1}{R(r)}r\frac{\partial^2}{\partial r^2}(r R(r))-l(l+1)+\frac{\omega^2}{c^2}r^2=0 \quad \text{where} \quad r^2\nabla_t^2 E_{i,t}=-l(l+1)E_{i,t}$$

The equation on the right just specifies the transverse behavior of the electric field and it is as far as we want to go with the transverse behavior. For the equation on the left, relabel $u=kr=\frac{\omega}{c}r$ to find:

$$\frac{1}{R(u)}u\frac{\partial^2}{\partial u^2}(u R(u))-l(l+1)+u^2=0$$

Now make the substitution $R(u)=U(u)/\sqrt{u}$ to find after much work:

$$u^2\frac{\partial^2 U}{\partial u^2}+u\frac{\partial U}{\partial u}+(u^2-(l+1/2)^2)U=0$$

This is Bessel's Differential Equation and its complex-valued solutions are again Hankel functions:

$$U(u)=a H_{l+1/2}^{(1)}(u)+b H_{l+1/2}^{(2)}(u)$$

which when substituting back $u=kr$ and $R(u)=U(u)/\sqrt{u}$ leads to:

$$R(r)=a\frac{H_{l+1/2}^{(1)}(kr)}{\sqrt{r}}+b\frac{H_{l+1/2}^{(2)}(kr)}{\sqrt{r}}$$

Again, for large *r* (far away from the origin), the Hankel functions asymptotically approach the forms:

$$H_m^{(1)}(x)=C\frac{1}{\sqrt{x}}e^{-ix} \quad \text{and} \quad H_m^{(2)}(x)=D\frac{1}{\sqrt{x}}e^{ix}$$

Therefore, for large r, the radial part of the electric field solution becomes:

$$R(r)=a\frac{e^{\pm ikr}}{r}$$

In summary, the particular solution to the wave equation for spherical wavefronts far from the origin is:

$$\boxed{E_i=E_{i,t}(\phi,\theta)\frac{e^{\pm ikr-i\omega t}}{r}} \quad \text{where } k=\frac{\omega}{c} \text{ and } E_{i,t} \text{ is real and obeys } r^2\nabla_t^2 E_{i,t}=-l(l+1)E_{i,t}$$

These equations tell us that the particular solution to the wave equation with spherical wavefronts far away from the origin is sinusoidal traveling waves that are traveling outwards or inwards in the radial direction. We call these waves "spherical waves" since on a given spherical surface centered on the origin, the electric field has constant phase across the surface. Conceptually, spherical waves are created by localized oscillating charge distributions. The factor of $1/r$ is needed in the above equation to ensure conservation of energy. Since the total energy carried along by an expanding wavefront must be constant, and since the total energy is the energy density times the area, the energy density must decrease at the rate that the sphere's area is increasing. Since a sphere's area is $4\pi r^2$, and thus increases quadratically with radius, the energy density must be proportional to $1/r^2$. Because the electric field is proportional to the square root of the energy density, the electric field strength must be proportional to. $1/r$. Thus we see that energy conservation is the explanation of the appearance of this factor in the solution above.