Lecture 5 Notes, Electromagnetic Theory II

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1. Waveguides Continued

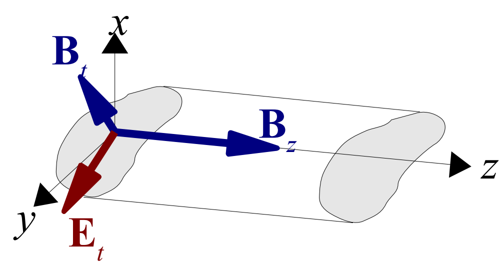
- In the previous lecture we made the assumption that the wave is harmonic in time and in the z dimension, but has more complicated dependence on the transverse dimensions.
- From this simple set-up, we were able to find the transverse fields in terms of the axial fields:

$$\mathbf{E}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\left(k\nabla_t E_z - \omega\hat{\mathbf{z}}\times\nabla_t B_z\right) \qquad \mathbf{B}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\left(k\nabla_t B_z + \mu\epsilon\omega\hat{\mathbf{z}}\times\nabla_t E_z\right)$$

- We then broke these two equations into TE and TM modes:

TRANSVERSE ELECTRIC (TE) WAVE:

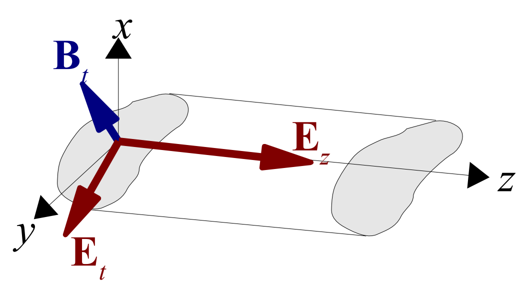
$$\boxed{\mathbf{E}_t = -i\frac{\omega}{\kappa^2}\left(\hat{\mathbf{z}}\times\nabla_t B_z\right)} \qquad \boxed{\mathbf{B}_t = i\frac{k}{\kappa^2}\left(\nabla_t B_z\right)}$$



TRANSVERSE MAGNETIC (TM) WAVE:

$$\boxed{\mathbf{E}_t = i\frac{k}{\kappa^2}\left(\nabla_t E_z\right)} \qquad \boxed{\mathbf{B}_t = i\frac{\mu\epsilon\omega}{\kappa^2}\left(\hat{\mathbf{z}}\times\nabla_t E_z\right)}$$

where $\kappa^2 = \epsilon\mu\omega^2 - k^2$



- There is a special case where both the electric field and the magnetic field are completely transverse, i.e. $B_z = 0$ and $E_z = 0$. This is called a transverse electromagnetic (TEM) wave.
- A TEM wave is not supported in simple hollow waveguides. A plane wave in free space or in an infinite uniform material exists in the TEM_{00} mode. TEM_{00} waves are also the dominant modes in coaxial cables.
- The problem has been reduced to finding the axial components of the fields, since we can then use the waveguide equations to find the transverse components.
- Start with the general waveguide equation:

$$\nabla^2\mathbf{E} - \mu\epsilon\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$

- This equation is a vector equation and therefore actually represents three different equations, one for each of the vector components. All we care about is the axial component:

$$\nabla^2 E_z - \mu\epsilon \frac{\partial^2 E_z}{\partial t^2} = 0$$

$$\nabla_t^2 E_z + \frac{\partial^2 E_z}{\partial z^2} - \mu\epsilon \frac{\partial^2 E_z}{\partial t^2} = 0$$

- Since we already specified the time and *z* dependence as harmonic, we can evaluate these derivatives:

$$\nabla_t^2 E_z - k^2 E_z + \mu\epsilon\omega^2 E_z = 0$$

$$\nabla_t^2 E_z = -(\mu\epsilon\omega^2 - k^2) E_z$$

$$\boxed{\nabla_t^2 E_z = -\kappa^2 E_z}$$

- This is the equation that needs to be solved to find the TM waves of the waveguide. We do this by applying boundary conditions.
- We can repeat the exact same approach with the magnetic wave equation and end up with the equation that needs to be solved to find the TE modes:

$$\boxed{\nabla_t^2 B_z = -\kappa^2 B_z}$$

- The particular solutions to these equations correspond to different κ_i where i = 1, 2, 3 and are known as the "modes" that the waveguide supports.
- The lowest-order modes are the first ones to appear as we slowly increase the amount of electromagnetic wave energy we are inserting into the waveguide.
- Often having the wave propagate in only the lowest mode is desirable. The waveguide dimensions and wave frequency can be chosen to ensure this.
- All of the results thus far are completely general and apply to *any* waveguide that is uniform along its axis. The specific choice of waveguide shape and materials are what specify the boundary conditions and material parameters.
- If we are assuming perfectly conducting walls, then there are no electric fields parallel to the wall surfaces and no magnetic fields normal to the surfaces.
- In the standard waveguide coordinate system, perfectly conducting walls exert the boundary conditions:

$$\boxed{[E_z]_{\text{on } S} = 0} \quad \text{and} \quad \boxed{\left[\frac{\partial B_z}{\partial n}\right]_{\text{on } S} = 0}$$

- If we look at the waveguide dispersion relation:

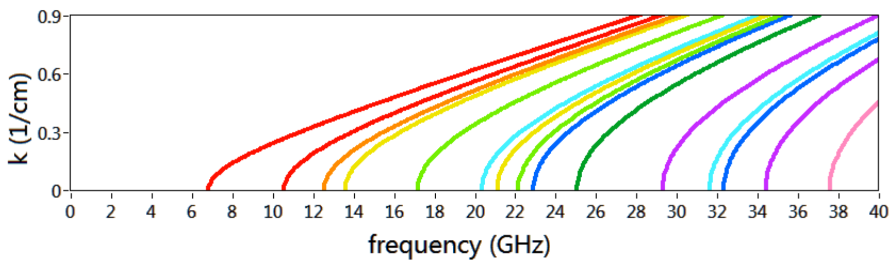
$$k = \sqrt{\epsilon\mu\omega^2 - \kappa^2}$$

we see that there is a minimum frequency below which waves cannot propagate. To find this "cutoff frequency", simply set $k = 0$ and solve for the frequency:

$$0 = \sqrt{\epsilon\mu\,\omega^2 - \kappa^2}$$

$$\boxed{\omega_m = \frac{\kappa_m}{\sqrt{\mu\,\epsilon}}}$$ *Waveguide Cutoff Frequency for Mode m*

- Above the cutoff frequency, regular traveling waves can propagate down the waveguide. Below the cutoff frequency, waves cannot travel down the waveguide and we instead get evanescent waves.
- Note that each mode has its own cutoff frequency. In general, higher-order modes have higher transverse wavenumbers and therefore higher cutoff frequencies.
- To make more sense of this, we plot the waveguide dispersion relation for TE modes in a hollow rectangular metal waveguide that is 14 cm wide and 9 cm tall:



- From this plot we see that at high frequencies, the curves approach the linear relationship characteristic of the free-space dispersion relation. Therefore, at very high frequencies, the waves simply act like waves in free space.
- We also see that the cutoff frequencies (where $k = 0$) of the higher modes (the blue and violet lines) are far above the cutoff frequencies of the lowest modes. This means, for instance, that if I send a 15 GHz TE wave into this waveguide, I can only excite the first four modes.
- For this waveguide, purely single-mode operation exists between about 7 and 10 GHz.

2. Rectangular Waveguides

- Consider a rectangular waveguide with perfect conducting walls, where one corner is placed at the origin and the other corner is placed at ($x = a$, $y = b$).
- Let us solve for TM waves as an example and leave the TE case to the interested reader.
- Expand the transverse wave equation into rectangular coordinates:

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} = -\kappa^2 E_z$$

- The boundary condition that applies to this case specifies that the field E_z must vanish everywhere on the boundary, which is at $x = 0$, $x = a$, $y = 0$, and $y = b$.
- The solution therefore has the form: $E_z = E_0 \sin(A\,x)\sin(B\,y)$.
- Plug this trial solution into the transverse wave equation to find:

$$-A^2 E_0 \sin(A\,x)\sin(B\,y) - B^2 E_0 \sin(A\,x)\sin(B\,y) = -\kappa^2 E_0 \sin(A\,x)\sin(B\,y)$$

$$\kappa^2 = A^2 + B^2$$

- Apply the boundary condition $E_z(x = a, y) = 0$:

$$0 = E_0 \sin(A a) \sin(B y)$$

$$A = \frac{m\pi}{a} \quad \text{where } m = 0, 1, 2, \ldots$$

- Similarly, the other boundary condition leads to:

$$B = \frac{n\pi}{b} \quad \text{where } n = 0, 1, 2, \ldots$$

- We have our final solution, remembering to include the z dependence and time dependence:

$$\boxed{E_z = E_0 \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{ikz - i\omega t}}$$

Rectangular Waveguide TM Solution

where $\boxed{\kappa^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}$, $\boxed{k^2 = \mu\epsilon\omega^2 - \kappa^2}$, $m = 0, 1, 2, \ldots$ and $n = 0, 1, 2, \ldots$

- With rectangular waveguides, the modes are typically labeled TM_{mn}.
- For instance, the transverse magnetic mode where $m = 2$ and $n = 1$ is written as TM_{21} and the corresponding axial electric field is:

$$E_{z,21} = E_0 \sin\left(\frac{2\pi x}{a}\right) \sin\left(\frac{\pi y}{b}\right) e^{ikz - i\omega t} \quad \text{where} \quad k = \sqrt{\mu\epsilon\omega^2 - \left(\frac{4\pi^2}{a^2} + \frac{\pi^2}{b^2}\right)}$$

- Note that if I choose $m = 0$ or $n = 0$ for TM modes, then the boxed equation above tells me that E_z is always zero. The waveguide equations therefore tell me that all fields components are zero, which is just the trivial solution. Therefore, any TM mode of a hollow rectangular waveguide with $m = 0$ or $n = 0$ is not a valid mode. Modes such as TM_{00}, TM_{10}, TM_{01}, TM_{02}, etc are not valid modes.
- If we repeat this type of approach for TE modes, we find the solution to be:

$$\boxed{B_z = B_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{ikz - i\omega t}}$$

Rectangular Waveguide TE Solution

- Note that if we choose $m = 0$ or $n = 0$ for TE modes then B_z is still a valid mode pattern. Therefore, these are valid modes. However, in the special case of $m = 0$ **and** $n = 0$, B_z becomes a constant which when inserted into the waveguide equations reveals that all other field components are zero. Therefore, TE_{00} is not a valid mode.

3. Energy Flow in Waveguides

- Let us calculate the Poynting vector for the fields inside an axially uniform waveguide.

$$\mathbf{S} = (\Re(\mathbf{E})) \times (\Re(\mathbf{H}))$$

- Before doing this, let us try to get the real-part operator outside so that we don't have to worry about it until the end.

$$\mathbf{S}=\frac{1}{4}(\mathbf{E}+\mathbf{E}^*)\times(\mathbf{H}+\mathbf{H}^*)$$

$$\mathbf{S}=\frac{1}{4}(\mathbf{E}\times\mathbf{H}+\mathbf{E}\times\mathbf{H}^*+\mathbf{E}^*\times\mathbf{H}+\mathbf{E}^*\times\mathbf{H}^*)$$

$$\boxed{\mathbf{S}=\frac{1}{2}\Re(\mathbf{E}\times\mathbf{H}^*+\mathbf{E}\times\mathbf{H})}$$

- This expression is still completely general.
- For fields that vary harmonically in time, the time-average of the second term vanishes, leaving:

$$\boxed{\langle\mathbf{S}\rangle=\frac{1}{2}\Re(\mathbf{E}\times\mathbf{H}^*)}$$ *Time-averaged Poynting vector for harmonic fields*

- Expand the fields in the equation above into transverse and axial components:

$$\langle\mathbf{S}\rangle=\frac{1}{2\mu}\Re[(\mathbf{E}_t+\hat{\mathbf{z}}E_z)\times(\mathbf{B}_t^*+\hat{\mathbf{z}}B_z^*)]$$

- Break this into TE ($E_z = 0$) and TM ($B_z = 0$) modes:

TE: $\langle\mathbf{S}\rangle=\frac{1}{2\mu}\Re[\mathbf{E}_t\times(\mathbf{B}_t^*+\hat{\mathbf{z}}B_z^*)]$ TM: $\langle\mathbf{S}\rangle=\frac{1}{2\mu}\Re[(\mathbf{E}_t+\hat{\mathbf{z}}E_z)\times\mathbf{B}_t^*]$

- If we apply the waveguide equations to remove the transverse fields, we find:

$$\boxed{\langle\mathbf{S}\rangle=\Re\left[\frac{1}{\mu}\frac{\omega k}{2\kappa^4}\left(\hat{\mathbf{z}}|\nabla_t B_z|^2-i\frac{\kappa^2}{k}B_z^*\nabla_t B_z\right)\right]}$$ for TE modes

$$\boxed{\langle\mathbf{S}\rangle=\Re\left[\epsilon\frac{\omega k}{2\kappa^4}\left(\hat{\mathbf{z}}|\nabla_t E_z|^2+i\frac{\kappa^2}{k}E_z\nabla_t E_z^*\right)\right]}$$ for TM modes

- The first term in each case is the axial energy flow and the second term in each case is the transverse energy flow.
- The transverse Poynting vector just represents the energy bouncing back and forth between the walls and has little practical importance.
- If we only care about the power guided down the waveguide, we can dot both sides with the axial unit vector:

$$\boxed{\langle \mathbf{S} \rangle \cdot \hat{\mathbf{z}} = \Re\left[\frac{1}{\mu}\frac{\omega k}{2\kappa^4}|\nabla_t B_z|^2\right]}$$ for TE modes

$$\boxed{\langle \mathbf{S} \rangle \cdot \hat{\mathbf{z}} = \Re\left[\epsilon\frac{\omega k}{2\kappa^4}|\nabla_t E_z|^2\right]}$$ for TM modes

- Note that the energy is predominantly guided at the points in space where the axial field has a high gradient, such as at the nodal lines.
- Further considerations about the energy flow in waveguides can be found in the Appendix.

4. Resonant Cavities and Lasers

- A resonant cavity is an object in which *every* component of the electromagnetic field is bound and resonates as a standing wave.
- In the waveguide, the wave resonated in the transverse directions, but propagated freely along the waveguide's axis. As a result, *any frequency* was allowed above the cutoff frequency for a given mode.
- In a cavity, however, the wave resonates in all directions. As a result, waves can only exist at the *resonant frequencies of the cavity*.
- Any hollow object of any shape with electromagnetically reflective walls is a resonant cavity.
- The most useful resonant cavities, however, are the ones made by taking a uniform waveguide and placing flat reflective surfaces on the ends.
- For cavities, the axial wavenumber k is no longer a free parameter dependent on our choice of wave frequency. Rather it is a fixed property of the waveguide geometry for a given mode and forces the frequency of a certain mode to be a single value.
- Assume that the ends are flat and perpendicular to the waveguide's axis, and that they are made of perfectly conducting material.
- Place one end at $z = 0$ and the other at $z = d$. Since the electric field must be normal and the magnetic field must be parallel to the surface of a perfect conductor, the boundary conditions are:

$$\mathbf{E}_t = 0 \text{ and } B_z = 0 \text{ at } z = 0 \text{ and } z = a$$

- Application of these boundary conditions leads to:

$$\mathbf{E}_t(x, y, z, t) = \mathbf{E}_t(x, y, t)\sin(kz) \qquad \mathbf{B}_t(x, y, z, t) = \mathbf{B}_t(x, y, t)\cos(kz)$$

$$E_z(x, y, z, t) = E_z(x, y, t)\cos(kz) \qquad B_z(x, y, z, t) = B_z(x, y, t)\sin(kz)$$

where $k = \frac{p\pi}{d}$ and p = 0, 1, 2, ...

- With the axial wavenumber fixed, the frequency now becomes fixed:

$$\kappa^2 = \mu\epsilon\omega^2 - k^2$$

$$\omega = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\kappa^2 + k^2}$$

$$\omega_{\kappa p}=\frac{1}{\sqrt{\mu\epsilon}}\sqrt{\kappa^2+\frac{p^2\pi^2}{d^2}}$$

$$\boxed{\omega_{mnp}=\frac{1}{\sqrt{\mu\epsilon}}\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}+\frac{p^2\pi^2}{d^2}}}$$

Resonant Frequencies of a Rectangular Waveguide

- Lasers are typically built by placing an amplifying medium in a resonant cavity.
- The modes that a laser emits depends on the geometry of its cavity.
- In order for the laser to emit some of the wave out of the device, the end surfaces should not be 100% reflective. Typically though, they are > 90% reflective and treating the cavity as an ideal resonant cavity is a good approximation.
- When a resonant cavity is made up of flat ends that are not 100% reflective, it is known as a "Fabry-Perot Cavity".

5. Quality of a Cavity or Laser

- So far we have only discussed lossless dielectric material inside the cavity and lossless perfect conductors forming the walls. What happens if we include materials that have loss?
- Mathematically, many of the previous equations are still used, but we understand now that the permittivity and wave numbers are complex-valued numbers.
- The "*Q*-factor", or "quality" of a cavity measures the cavity's ability to store oscillating electromagnetic energy (i.e. the ability to not lose oscillating energy to the materials).
- The *Q* factor is the normalized lifetime of the decaying wave:

$$\boxed{Q=2\pi\frac{\tau}{T}}$$

where τ is the lifetime of the exponentially-decaying energy of the wave (as it loses energy to the cavity) and T is the period of the wave's oscillation.
- We have divided by T to get a dimensionless number that is representative of the cavity no matter what frequency is used.
- Instead of time scales, we can use inverse time scales:

$$\boxed{Q=\frac{\omega}{\lambda}}$$

where ω is the frequency of the wave and λ is the decay rate at which it loses energy.
- The *Q* factor can also be thought of as frequency times the ratio of stored energy to rate of energy loss:

$$Q=\omega\frac{U}{-dU/dt}$$

- Let us show that this is equivalent to the first definition.

$$\frac{dU}{dt}=-\omega\frac{U}{Q}$$

- This differential equation has the solution:

$$U = U_0 e^{-\omega t/Q}$$

- Comparing this to the general form of an exponentially-decaying quantity, $U = U_0 e^{-t/\tau}$ (where τ is the lifetime), we can equate the lifetime to the Q-factor:

$$\tau = \frac{Q}{\omega} \quad \text{or} \quad \tau = \frac{T}{2\pi} Q \quad \text{so that} \quad Q = 2\pi \frac{\tau}{T}$$

- This just shows that this definition for the Q-factor is equivalent to the first definition.
- A wave that is decaying cannot be represented by a single frequency.
- The loss of energy means that instead of a single frequency being associated with a particular mode, there is a narrow range of possible frequencies associated with a particular mode. Let us investigate this fact mathematically.
- The time dependence of the electric field, when including exponential time decay, becomes:

$$E(t) = E_0 e^{-t/2\tau} e^{-i\omega_0 t}$$

$$E(t) = E_0 e^{-\omega_0 t/2Q} e^{-i\omega_0 t}$$

where ω_0 is the resonant frequency.
- Let us Fourier transform the equation above into frequency space to see what the spectrum of the wave looks like:

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty E(t) e^{i\omega t}\, dt$$

$$E(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty E_0 e^{-\omega_0 t/2Q} e^{-i\omega_0 t} e^{i\omega t}\, dt$$

$$E(\omega) = \frac{1}{\sqrt{2\pi}} E_0 \int_0^\infty e^{(i(\omega-\omega_0)-\omega_0/2Q)t}\, dt$$

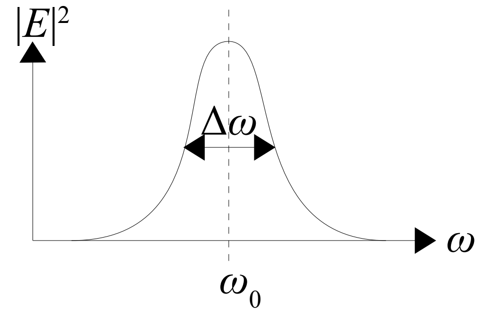
$$E(\omega) = \frac{1}{\sqrt{2\pi}} E_0 \left[\frac{e^{(i(\omega-\omega_0)-\omega_0/2Q)t}}{(i(\omega-\omega_0)-\omega_0/2Q)} \right]_0^\infty$$

$$E(\omega) = \frac{1}{\sqrt{2\pi}} E_0 \left[\frac{1}{(i(\omega-\omega_0)-\omega_0/2Q)} \right]$$

- The intensity is proportional to the magnitude squared of the electric field:

$$|E(\omega)|^2 = \left| \frac{1}{\sqrt{2\pi}} E_0 \left[\frac{1}{(i(\omega-\omega_0)-\omega_0/2Q)} \right] \right|^2$$

$$\boxed{|E(\omega)|^2 = \frac{1}{2\pi}|E_0|^2 \frac{1}{(\omega-\omega_0)^2+(\omega_0/2Q)^2}}$$



- This is known as the lineshape of the radiation supported by the cavity. It is a Lorentzian.
- A representative Lorentzian lineshape is plotted on the right.
- The peak of this curve exists at $\omega = \omega_0$. Plugging this into the equation above, we find:

$$|E(\omega_0)|^2 = \frac{2}{\pi}|E_0|^2 \frac{Q^2}{\omega_0^2}$$

- We see that for a cavity with a high Q-factor, more of the energy is concentrated at the peak frequency.
- The full width of the curve at half of its height $\Delta\omega$ is known as the spectral linewidth of the cavity. It is also called the Full-Width at Half-Max (FWHM) and labeled with the letter Γ.
- To find the spectral linewidth, we simply set the boxed equation above equal to half its peak value and solve for the frequency.

$$\frac{1}{2\pi}|E_0|^2 \frac{1}{(\omega-\omega_0)^2+(\omega_0/2Q)^2} = \frac{1}{2}|E(\omega_0)|^2$$

$$\frac{1}{2\pi}|E_0|^2 \frac{1}{(\omega-\omega_0)^2+(\omega_0/2Q)^2} = \frac{1}{2}\left[\frac{2}{\pi}|E_0|^2 \frac{Q^2}{\omega_0^2}\right]$$

$$\omega_1 = \omega_0 - \frac{\omega_0}{2Q} \quad \text{and} \quad \omega_2 = \omega_0 + \frac{\omega_0}{2Q}$$

- Therefore, the spectral linewidth can be found:

$$\Delta\omega = \omega_2 - \omega_1$$

$$\boxed{\Delta\omega = \frac{\omega_0}{Q}}$$

- We can also invert this equation to find another definition for the Q-factor.

$$\boxed{Q = \frac{\omega_0}{\Delta\omega}}$$

- These equations tells us that a higher Q, which corresponds to less energy loss, means a narrower range of frequencies in the wave of a single mode.
- For lasers, this means that the spectral width of the laser beam is narrower for cavities with less loss, which is usually desirable for spectroscopy as it gives you better resolution.

APPENDIX

We can derive some interesting properties for the fields inside waveguides using the general waveguide equations. The total fields in a waveguide in terms of the parallel components are:

TE: $\mathbf{E}=-i\frac{\omega}{\kappa^2}\hat{\mathbf{z}}\times\nabla_t B_z$ and $\mathbf{B}=B_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}\nabla_t B_z$

TM: $\mathbf{E}=E_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}\nabla_t E_z$ and $\mathbf{B}=i\frac{\mu\epsilon\omega}{\kappa^2}\hat{\mathbf{z}}\times\nabla_t E_z$

Note that if we take the equations above for the TE modes and relabel every field component according to $\mathbf{B}\rightarrow\mathbf{E}$ and $\mathbf{E}\rightarrow-\mathbf{B}/\mu\epsilon$, then we end up with the equations:

$$\mathbf{B}=i\frac{\omega\mu\epsilon}{\kappa^2}\hat{\mathbf{z}}\times\nabla_t E_z \quad \text{and} \quad \mathbf{E}=E_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}\nabla_t E_z$$

These are the exact same equations as for the TM mode. Thus, if we have found a relationship between the fields of the TE modes of a particular waveguide, we immediately also know the TM mode relationships by making the substitutions $\mathbf{B}\rightarrow\mathbf{E}$ and $\mathbf{E}\rightarrow-\mathbf{B}/\mu\epsilon$. Since all we did was use the general waveguide equations, this trick works for all waveguides that are uniform along their axis.

Similarly, if we take the equations for the total fields of the TM modes and make the substitutions $\mathbf{B}\rightarrow-\mu\epsilon\mathbf{E}$ and $\mathbf{E}\rightarrow\mathbf{B}$, then we get the equations for the TE modes. These results can be summarized as:

Mode Symmetry Relations		
$TE\rightarrow TM$ if	$\mathbf{B}\rightarrow\mathbf{E}$	$\mathbf{E}\rightarrow-\mathbf{B}/\mu\epsilon$
$TM\rightarrow TE$ if	$\mathbf{B}\rightarrow-\mu\epsilon\mathbf{E}$	$\mathbf{E}\rightarrow\mathbf{B}$

A separate property can be derived by taking the dot product of the total electric and magnetic field in a waveguide

TE: $\mathbf{E}\cdot\mathbf{B}=\left[-i\frac{\omega}{\kappa^2}\hat{\mathbf{z}}\times\nabla_t B_z\right]\cdot\left[B_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}\nabla_t B_z\right]$ TM: $\mathbf{E}\cdot\mathbf{B}=\left[E_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}\nabla_t E_z\right]\cdot\left[i\frac{\mu\epsilon\omega}{\kappa^2}\hat{\mathbf{z}}\times\nabla_t E_z\right]$

For the TE mode, the electric field lies in the transverse plane and thus is perpendicular to the *z* direction by definition, meaning that the dot product of the electric field and the parallel magnetic field is zero. Similarly, for the TM mode, the dot product of the magnetic field and the parallel electric field is zero. This leaves:

TE: $\mathbf{E}\cdot\mathbf{B}=\frac{\omega k}{\kappa^4}\left[\hat{\mathbf{z}}\times\nabla_t B_z\right]\cdot\left[\nabla_t B_z\right]$ TM: $\mathbf{E}\cdot\mathbf{B}=-\frac{k}{\kappa^4}\left[\nabla_t E_z\right]\cdot\left[\hat{\mathbf{z}}\times\nabla_t E_z\right]$

Next note that both equations have the mathematical form $\left[\hat{\mathbf{z}}\times\mathbf{A}\right]\cdot\mathbf{A}$. Using geometry, or by expanding this expression into rectangular components, it is easy to show that this expression equals zero.

Therefore, we have:

$$\boxed{\text{TE: } \mathbf{E}\cdot\mathbf{B}=0, \text{ TM: } \mathbf{E}\cdot\mathbf{B}=0}$$

This result tells us that the total electric field and total magnetic field are always perpendicular in a waveguide for a given mode. Note that this derivation only holds true for a single mode. We can of course get the total **E** and **B** fields to not be perpendicular if we add together several different modes.

Similarly, if we take the dot product of the total electric field and the total wavevector, $\mathbf{k}_{\text{tot}}=\mathbf{k}+\boldsymbol{\kappa}$, we get:

$$\text{TE:}\quad \mathbf{E}\cdot\mathbf{k}_{\text{tot}}=\left[-i\frac{\omega}{\kappa^2}\hat{\mathbf{z}}\times\nabla_t B_z\right]\cdot[\mathbf{k}+\boldsymbol{\kappa}]\qquad \text{TM:}\quad \mathbf{E}\cdot\mathbf{k}_{\text{tot}}=\left[E_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}\nabla_t E_z\right]\cdot[\mathbf{k}+\boldsymbol{\kappa}]$$

Since transverse components are always perpendicular to axial components, these expressions simplify to:

$$\text{TE:}\quad \mathbf{E}\cdot\mathbf{k}_{\text{tot}}=-i\frac{\omega}{\kappa^2}[\hat{\mathbf{z}}\times\nabla_t B_z]\cdot\boldsymbol{\kappa}\qquad \text{TM:}\quad \mathbf{E}\cdot\mathbf{k}_{\text{tot}}=k\,E_z+i\frac{k}{\kappa^2}[\nabla_t E_z]\cdot\boldsymbol{\kappa}$$

Expanding the transverse gradient into rectangular components and assuming that the fields have simple harmonic dependence in *x* and *y*, we can show that $\nabla_t B_z=i\boldsymbol{\kappa}B_z$ and $\nabla_t E_z=i\boldsymbol{\kappa}E_z$, leading to:

$$\text{TE:}\quad \mathbf{E}\cdot\mathbf{k}_{\text{tot}}=\frac{\omega}{\kappa^2}B_z[\hat{\mathbf{z}}\times\boldsymbol{\kappa}]\cdot\boldsymbol{\kappa}\qquad \text{TM:}\quad \mathbf{E}\cdot\mathbf{k}_{\text{tot}}=k\,E_z-\frac{k}{\kappa^2}E_z\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}$$

$$\boxed{\text{TE: } \mathbf{E}\cdot\mathbf{k}_{\text{tot}}=0, \text{ TM: } \mathbf{E}\cdot\mathbf{k}_{\text{tot}}=0}$$

We thus see that the total electric field is always perpendicular to the total wavevector in a waveguide.

Similarly, if we take the dot product of the total magnetic field and the total wavevector, we get:

$$\text{TE:}\quad \mathbf{B}\cdot\mathbf{k}_{\text{tot}}=\left[B_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}\nabla_t B_z\right]\cdot[\mathbf{k}+\boldsymbol{\kappa}]\qquad \text{TM:}\quad \mathbf{B}\cdot\mathbf{k}_{\text{tot}}=\left[i\frac{\mu\epsilon\omega}{\kappa^2}\hat{\mathbf{z}}\times\nabla_t E_z\right]\cdot[\mathbf{k}+\boldsymbol{\kappa}]$$

$$\text{TE:}\quad \mathbf{B}\cdot\mathbf{k}_{\text{tot}}=k\,B_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}[\nabla_t B_z]\cdot\boldsymbol{\kappa}\qquad \text{TM:}\quad \mathbf{B}\cdot\mathbf{k}_{\text{tot}}=i\frac{\mu\epsilon\omega}{\kappa^2}[\hat{\mathbf{z}}\times\nabla_t E_z]\cdot\boldsymbol{\kappa}$$

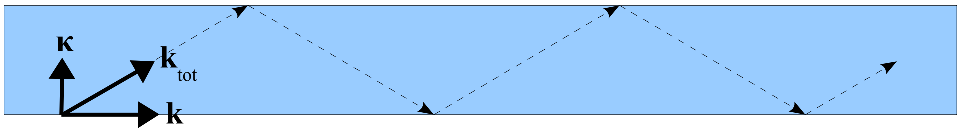
Applying $\nabla_t B_z=i\boldsymbol{\kappa}B_z$ and $\nabla_t E_z=i\boldsymbol{\kappa}E_z$, we get:

$$\text{TE:}\quad \mathbf{B}\cdot\mathbf{k}_{\text{tot}}=k\,B_z-\frac{k}{\kappa^2}B_z\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}\qquad \text{TM:}\quad \mathbf{B}\cdot\mathbf{k}_{\text{tot}}=-\frac{\mu\epsilon\omega}{\kappa^2}E_z[\hat{\mathbf{z}}\times\boldsymbol{\kappa}]\cdot\boldsymbol{\kappa}$$

$$\boxed{\text{TE: } \mathbf{B}\cdot\mathbf{k}_{\text{tot}}=0, \text{ TM: } \mathbf{B}\cdot\mathbf{k}_{\text{tot}}=0}$$

We see that the total magnetic field is always perpendicular to the total wavevector in a waveguide.

Therefore, the total electric field, the total magnetic field, and the total wavevector form a mutually-orthogonal triplet just like in a simple transverse wave. In fact, instead of thinking of the fields in a waveguide as complicated waves with axial components traveling straight down the waveguide axis, we can think of them as transverse waves traveling in the oblique direction $\mathbf{k}_{\text{tot}}=\mathbf{k}+\boldsymbol{\kappa}$, bouncing back and forth down the waveguide, as shown in the schematic diagram.



Taking the cross product of the total electric field and the total wavevector should confirm these concepts:

TE: $\mathbf{k}_{\text{tot}}\times\mathbf{E}=[\mathbf{k}+\boldsymbol{\kappa}]\times\left[-i\frac{\omega}{\kappa^2}\hat{\mathbf{z}}\times\nabla_t B_z\right]$ TM: $\mathbf{k}_{\text{tot}}\times\mathbf{E}=[\mathbf{k}+\boldsymbol{\kappa}]\times\left[E_z\hat{\mathbf{z}}+i\frac{k}{\kappa^2}\nabla_t E_z\right]$

$\boxed{\text{TE: } \mathbf{k}_{\text{tot}}\times\mathbf{E}=\omega\mathbf{B}, \text{ TM: } \mathbf{k}_{\text{tot}}\times\mathbf{E}=\omega\mathbf{B}}$

These results confirm that the **E**, **B**, and $\mathbf{k}_{\text{tot}}$ vectors form a right-handed orthogonal triple just like in a plane wave.

The total power P flowing down the waveguide is the integral of the Poynting vector over the entire cross-sectional area of the waveguide:

$$P=\int_A <\mathbf{S}>\cdot\mathbf{z}\,da$$

$$P=\Re\left[\frac{1}{\mu}\frac{\omega k}{2\kappa^4}\int_A|\nabla_t B_z|^2\,da\right] \quad \text{for TE modes}$$

$$P=\Re\left[\epsilon\frac{\omega k}{2\kappa^4}\int_A|\nabla_t E_z|^2\,da\right] \quad \text{for TM modes}$$

Use Green's first identity in two dimensions, $\int_A|\nabla_t\psi|^2\,da=\oint_C\psi^*\frac{\partial\psi}{\partial n}dl-\int_A\psi^*\nabla_t^2\psi\,da$ to find:

$$P=\Re\left[\frac{1}{\mu}\frac{\omega k}{2\kappa^4}[\oint_C B_z^*\frac{\partial B_z}{\partial n}dl-\int_A B_z^*\nabla_t^2 B_z\,da]\right] \quad \text{for TE modes}$$

$$P=\Re\left[\epsilon\frac{\omega k}{2\kappa^4}[\oint_C E_z^*\frac{\partial E_z}{\partial n}dl-\int_A E_z^*\nabla_t^2 E_z\,da]\right] \quad \text{for TM modes}$$

The line integrals are over the boundary of the waveguide. For perfectly conducting walls, either the axial field or its derivative vanishes on the walls. Either way, the line integrals vanish. Also, the transverse wave equations (i.e. $\nabla_t^2 E_z=-\kappa^2 E_z$ and $\nabla_t^2 B_z=-\kappa^2 B_z$) can be used to transform the area integrals. These considerations reduce the power equations to:

$$\boxed{P=\Re\left[\frac{1}{\mu}\frac{\omega k}{2\kappa^2}\int_A |B_z|^2\,da\right]}$$ for TE modes

$$\boxed{P=\Re\left[\epsilon\frac{\omega k}{2\kappa^2}\int_A |E_z|^2\,da\right]}$$ for TM modes

Using these results we can also find the total energy stored in the fields, but we have to find the group velocity first. We found the dispersion relation in a waveguide previously to be $k^2=\mu\epsilon\omega^2-\kappa^2$ where the transverse wavenumber κ is a constant for a given mode. The group velocity is therefore:

$$v_g=\frac{d\omega}{dk}$$

$$v_g=\frac{d}{dk}\left(\sqrt{\frac{k^2+\kappa^2}{\mu\epsilon}}\right)$$

$$\boxed{v_g=\frac{k}{\omega\mu\epsilon}}$$

The total field energy per unit length U is the total power over the group velocity:

$$U=\frac{P}{v_g}$$

The total field energy per unit length is therefore:

$$\boxed{U=\Re\left[\frac{1}{\mu}\frac{\mu\epsilon\omega^2}{2\kappa^2}\int_A |B_z|^2\,da\right]}$$ for TE modes

$$\boxed{U=\Re\left[\epsilon\frac{\mu\epsilon\omega^2}{2\kappa^2}\int_A |E_z|^2\,da\right]}$$ for TM modes

For a rectangular waveguide, we can calculate these integrals. For the TE mode, the integral is:

$$\int_A |B_z|^2\,da=\int_0^b\int_0^a |B_z|^2\,dx\,dy$$

$$\int_A |B_z|^2\,da=|B_0|^2\int_0^a \cos^2\left(\frac{m\pi x}{a}\right)dx\int_0^b \cos^2\left(\frac{n\pi y}{b}\right)dy$$

$$\int_A |B_z|^2\,da=\frac{ab}{4h}|B_0|^2$$ where h = 1 if $m>0$ and $n>0$, h = ½ otherwise (TE_{00} is not a valid mode)

Inserting this into the expression for the power and energy we have:

$$\boxed{P = \Re\left[\frac{1}{\mu}\frac{\omega k\, a\, b}{8\mu\kappa^2 h}|B_0|^2\right]}$$

TE Power and Energy Inside a Rectangular Waveguide

$$\boxed{U = \Re\left[\frac{1}{\mu}\frac{\mu\epsilon\omega^2 a\, b}{8\kappa^2 h}|B_0|^2\right]}$$

where $\kappa^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$ and $k^2 = \mu\epsilon\omega^2 - \kappa^2$ and $h = 1$ if $m > 0$ and $n > 0$, $h = \frac{1}{2}$ otherwise

Similarly, we can do the integral for the TM mode:

$$\int_A |E_z|^2 da = \int_0^b \int_0^a |E_z|^2 dx\, dy$$

$$\int_A |E_z|^2 da = |E_0|^2 \int_0^a \sin^2\left(\frac{m\pi x}{a}\right) dx \int_0^b \sin^2\left(\frac{n\pi y}{b}\right) dy$$

$$\int_A |E_z|^2 da = \frac{a\, b}{4}|E_0|^2$$

Inserting this into the expression for the power and energy we have:

$$\boxed{P = \Re\left[\epsilon\frac{\omega k\, a\, b}{8\kappa^2}|E_0|^2\right]}$$

TM Power and Energy Inside a Rectangular Waveguide

$$\boxed{U = \Re\left[\epsilon\frac{\mu\epsilon\omega^2 a\, b}{8\kappa^2}|E_0|^2\right]}$$