Lecture 4 Notes, Electromagnetic Theory II

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1. Fields at the Surface of and Within a Conductor

- First consider a flat, *perfect* conductor bordering on a non-conducting medium.
- As in the electrostatics case, the charges in a perfect conductor react to the external fields fast enough to always cancel them out, so that there are no fields inside the conductor. The **E** field at the boundary surface is normal to the surface and the **H** field is parallel to the surface.
- Now consider a *good* conductor as opposed to a perfect conductor.
- The fields at and inside a good conductor must behave approximately as a perfect conductor.
- The fields will die off rapidly inside the conductor and will be attenuated exponentially according to some skin depth δ.
- The Maxwell-Ampere equation states:

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$$

- Apply this just inside the conductor, use Ohm's law $\mathbf{J} = \sigma\mathbf{E}$ and drop the **D** field because a good conductor involves conductive effects dominating over dielectric effects:

$$\boxed{\nabla\times\mathbf{H}_c=\sigma\,\mathbf{E}_c}$$

- Now take Faraday's Law:

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

- Apply this law just inside the conductor, and assume linear magnetic material and harmonic time dependence:

$$\boxed{\nabla\times\mathbf{E}_c=i\,\omega\mu_c\,\mathbf{H}_c}$$

- Since a good conductor is close to a perfect conductor, the fields must vary much more quickly in the direction into the conductor (call this the x direction) than in the other directions. Therefore, the gradient components in the other directions can be ignored:

$$\nabla=\mathbf{x}\frac{\partial}{\partial x}+\mathbf{y}\frac{\partial}{\partial y}+\mathbf{z}\frac{\partial}{\partial z}$$

$$\nabla\approx\mathbf{x}\frac{\partial}{\partial x}$$

- Define x such that it points into the conductor. Therefore, the conductor's surface normal (which points away from the conductor) is in the negative x direction.

$$\nabla \approx -\mathbf{n}\frac{\partial}{\partial x}$$

- Upon applying this gradient operator expression, the equations in boxes above become:

$$\mathbf{E}_c = -\frac{1}{\sigma}\mathbf{n}\times\frac{\partial \mathbf{H}_c}{\partial x}$$

$$\mathbf{H}_c = \frac{i}{\omega\mu_c}\mathbf{n}\times\frac{\partial \mathbf{E}_c}{\partial x}$$

- Combine these two equations to find:

$$\mathbf{H}_c = -\frac{i}{\mu_c\omega\sigma}\mathbf{n}\times\left(\mathbf{n}\times\frac{\partial^2 \mathbf{H}_c}{\partial x^2}\right)$$

- This is a vector equation. Let us split this equation into components normal to the surface and parallel to the surface.
- Dot both sides of the equation above with the surface normal vector to get the normal component of $\mathbf{H}_c$.

$$\mathbf{n}\cdot\mathbf{H}_c = -\frac{i}{\mu_c\omega\sigma}\mathbf{n}\cdot\left(\mathbf{n}\times\left(\mathbf{n}\times\frac{\partial^2 \mathbf{H}_c}{\partial x^2}\right)\right)$$

- The cross product of a vector **n** with something else will always be perpendicular to **n**, and the dot product of **n** with something perpendicular to it will always be zero, leading to:

$$\mathbf{n}\cdot\mathbf{H}_c = 0$$

- This is as we expected. There is no normal component of the magnetic field just outside the conductor, so there should not be one just inside the conductor. This result essentially follows from our assumption that the fields vary the most in the normal direction.
- Let us now find the component of the magnetic field parallel to the conductor's surface by crossing both sides with **n**:

$$\mathbf{n}\times\mathbf{H}_c = -\frac{i}{\mu_c\omega\sigma}\mathbf{n}\times\left(\mathbf{n}\times\left(\mathbf{n}\times\frac{\partial^2 \mathbf{H}_c}{\partial x^2}\right)\right)$$

- We now use the vector identity $\mathbf{n}\times(\mathbf{n}\times(\mathbf{n}\times\mathbf{A})) = -(\mathbf{n}\times\mathbf{A})$, which can be proved using simple geometry, to find:

$$\mathbf{n}\times\mathbf{H}_c = \frac{i}{\mu_c\omega\sigma}\frac{\partial^2}{\partial x^2}\mathbf{n}\times\mathbf{H}_c$$

- Let us call the parallel component $\mathbf{H}_{\mathrm{par}}$ so that $\mathbf{H}_{\mathrm{par}} = \mathbf{n}\times\mathbf{H}_c$ and rearrange the equation to find:

$$\frac{\partial^2}{\partial x^2}\mathbf{H}_{\text{par}} + i\mu_c\omega\sigma\mathbf{H}_{\text{par}} = 0$$

- Try a solution of the form $\mathbf{H}_{\text{par}} = \mathbf{H}_{\text{par},0}e^{ax}$:

$$a^2 + i\mu_c\omega\sigma = 0$$

- For the purpose of keeping the splitting up of real and imaginary parts simple, we will approximate that the conductivity and permeability are real-valued.

$$a = \sqrt{-i\mu_c\omega\sigma}$$

$$a = \sqrt{\mu_c\omega\sigma}\, e^{i3\pi/4}$$

$$a = -\sqrt{\mu_c\omega\sigma/2} + i\sqrt{\mu_c\omega\sigma/2}$$

- Plugging this expression for a back into the trial solution, we find:

$$\mathbf{H}_{\text{par}} = \mathbf{H}_{\text{par},0}e^{-\sqrt{\mu_c\omega\sigma/2}\,x}e^{i\sqrt{\mu_c\omega\sigma/2}\,x}$$

- We can recognize the skin depth as $\delta = \sqrt{2/(\mu_c\omega\sigma)}$ so that the final solution is:

$$\mathbf{H}_{\text{par}} = \mathbf{H}_{\text{par},0}e^{ix/\delta}e^{-x/\delta}$$

- For a finite conductivity, there is no surface current, so the parallel field **H** just outside the surface must equal the parallel field $\mathbf{H}_c$ just inside the surface.:

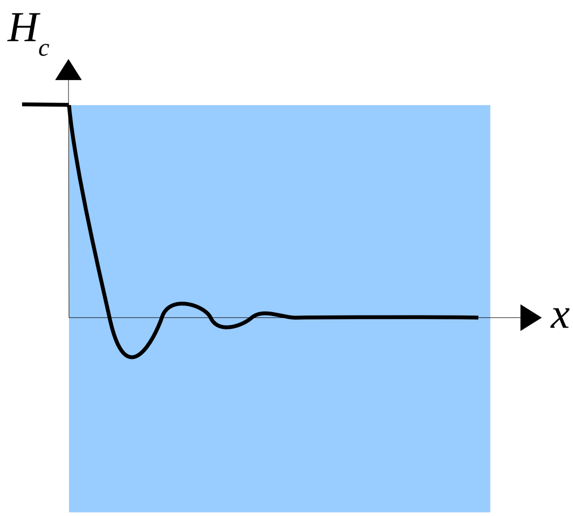
$$\mathbf{n}\times\mathbf{H} = \mathbf{n}\times\mathbf{H}_c \quad \text{at } x = 0$$

- Because there are no normal components of the magnetic field at the surface, the total field is is just the parallel field, indicating that the total magnetic field must be continuous across the conductor's surface. Therefore, we now know the total magnetic field inside a conductor in terms of the external magnetic field at its surface:

$$\boxed{\mathbf{H}_c = \mathbf{H}_{x=0}e^{-x/\delta}e^{i(x/\delta-\omega t)}} \quad \text{where} \quad \boxed{\delta = \sqrt{2/(\mu_c\omega\sigma)}}$$

Magnetic Field Inside a Good Conductor

- As expected, the field decays exponentially into the conductor. It also oscillates in time and space, indicating that we have a sinusoidal traveling wave with a wavenumber $k = 1/\delta$.
- Looking at the skin depth equation, we find that lower frequencies penetrate deeper into conductors. Also, the higher the conductivity, the lower the skin depth.
- We can plot this field:



- We can now plug the solution for the magnetic field back into the electric field equation to find:

$$\mathbf{E}_c = -\frac{1}{\sigma}\mathbf{n}\times\frac{\partial \mathbf{H}_c}{\partial x}$$

$$\mathbf{E}_c = -\frac{1}{\sigma}\mathbf{n}\times\frac{\partial}{\partial x}\mathbf{H}_{x=0}\, e^{i(x/\delta-\omega t)}e^{-x/\delta}$$

$$\mathbf{E}_c = (1-i)\frac{1}{\sigma\delta}\mathbf{n}\times\mathbf{H}_{x=0}\, e^{i(x/\delta-\omega t)}e^{-x/\delta}$$

$$\boxed{\mathbf{E}_c = \sqrt{\frac{\mu_c\omega}{\sigma}}\,\mathbf{n}\times\mathbf{H}_{x=0}\, e^{i(x/\delta-\omega t+7\pi/4)}e^{-x/\delta}}$$

- This solution tells us that the electric field inside a good conductor also forms traveling waves that decay exponentially into the conductor in the same way as the magnetic field.
- Note that the electric field is 45 degrees out of phase with the magnetic field.
- Also note that the electric field is perpendicular to the magnetic field, but is also parallel to the conductor's surface. Therefore all the fields in a conductor are parallel to the surface (at least in our approximation). The fields inside a good conductor are therefore strongly-attenuated transverse traveling waves propagating directly into the conductor.

2. Introduction to Waveguides

- Electrodynamics is essentially the study of the creation, propagation, and absorption of electromagnetic waves as they interact with matter.
- We have looked at the propagation of waves in free space, in bulk regions of uniform material and when passing through the boundary between two materials.
- We now move on to the next level of complexity: waves propagating along a waveguide.
- In general, a waveguide is a structure built by appropriately placing materials such that a wave is bound inside a structure and guided along its axis.
- There are two fundamental elements that must be specified before a waveguide can be

designed or analyzed: the method of containing the wave within the waveguide, and the cross-sectional shape of the waveguide.
- A wave is contained in the waveguide by making the walls (or areas external to the core) highly reflective so that the wave is continually reflected back into the core area.
- There are four main ways to reflect the wave:
  (1) Use conductors for the walls. This is especially useful for low frequencies such as in radar and electronics because the surface roughness of the walls is small compared to the wavelength and because the walls do not have to be fabricated at small dimensions.
  (2) Use plasma for the walls, such as in a highly doped semiconductor. This behaves similar to a metal, except that reflectivity can be custom-built by varying the doping. This is useful for solid state devices because the waveguide can be grown as part of the device. Gaseous plasmas can also be used. For instance, the earth's ionosphere acts as part of a waveguide for certain radio frequencies and can be exploited for long-range communications.
  (3) Use dielectric cladding with a lower index of refraction to take advantage of total internal reflection. Total internal reflection requires grazing angles of incidence, which is accomplished by shrinking the cross-sectional area of the waveguide. Dielectric waveguides are typically used at higher frequencies. For example, infrared and visible light signals are often guided using dielectric optical fibers.
  (4) Use dielectric material where holes have been formed in the outer region. Using interference effects, a photonic crystal is formed that is reflective at the desired frequency.
- The most common cross-sectional shapes for waveguides are rectangular and circular.

3. The Waveguide Equations

- Consider a waveguide where the fields inside have a harmonic time dependence, and the cross-sectional shape of the waveguide is constant along its axis.
- The guided wave is free along the waveguide's axis (the parallel direction). Mathematically, this means that the total fields have the forms:

$$\mathbf{E}(x, y, z, t) = (\mathbf{E}_t(x, y) + E_z(x, y)\hat{\mathbf{z}})e^{ikz - i\omega t}$$

$$\mathbf{B}(x, y, z, t) = (\mathbf{B}_t(x, y) + B_z(x, y)\hat{\mathbf{z}})e^{ikz - i\omega t}$$

- The general solution is the sum of all possible particular solutions, weighted by coefficients.
- These equations represent waves traveling down the waveguide in the z direction, with their x and y dependence being affected by the presence of the walls. Note that we have split the fields into components transverse to the waveguide axis (with subscript t), and components parallel to the waveguide axis (with subscript z).
- We note in advance that because we know the z and t dependence of the fields, we automatically know:

$$\frac{\partial \mathbf{E}}{\partial z} = ik\mathbf{E} \quad , \quad \frac{\partial \mathbf{B}}{\partial z} = ik\mathbf{B} \quad , \quad \frac{\partial \mathbf{E}}{\partial t} = -i\omega\mathbf{E} \quad \text{, and} \quad \frac{\partial \mathbf{B}}{\partial t} = -i\omega\mathbf{B}$$

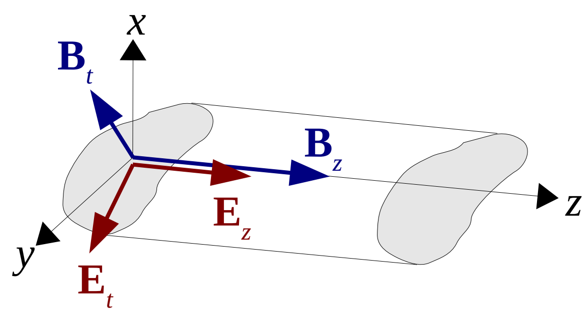
- Here, k represents the axial wavenumber, describing the spacing of wave peaks along the waveguide's axis. But we do not yet know how it relates to frequency since we are starting from Maxwell's equations and not assuming anything from our knowledge of infinite plane waves, which behave differently than guided waves.

- It is important to remember that vector fields have components in each coordinate dimension, *and* each component can be a function of all coordinate dimensions. We should not confuse components and functional dependence.
- Our task is to simplify Maxwell's equations as much as possible using the forms above and derive the transverse fields $\mathbf{E}_t$ and $\mathbf{B}_t$ in terms of the known parallel fields E_z and B_z, and the other way around as well.
- The derivation is a bit heavy on the math, so let us state some useful identities out front:

$$\nabla = \nabla_t + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

$$\mathbf{z}\times(\mathbf{z}\times\mathbf{B}_t) = -\mathbf{B}_t \quad , \quad \mathbf{z}\times(\mathbf{z}\times\mathbf{E}_t) = -\mathbf{E}_t$$

$$\mathbf{z}\times(\nabla_t\times\mathbf{B}_z) = \nabla_t B_z \quad , \quad \mathbf{z}\times(\nabla_t\times\mathbf{E}_z) = \nabla_t E_z$$



- Now let us start with Maxwell's equations without free currents or charges:

$$\nabla\cdot\mathbf{D} = 0 \quad , \qquad \nabla\cdot\mathbf{B} = 0 \quad ,$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad \text{, and} \quad \nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}$$

- Assume harmonic time dependence (this assumption places no restrictions on us because in the end, we can sum over all possible frequencies to get the general solution valid for any type of time dependence):

$$\nabla\cdot\mathbf{E} = 0 \quad , \qquad \nabla\cdot\mathbf{B} = 0 \quad ,$$

$$\nabla\times\mathbf{E} = i\omega\mathbf{B} \quad \text{, and} \quad \nabla\times\mathbf{B} = -i\omega\mu\epsilon\mathbf{E}$$

- Expanding the fields and gradients into parallel and transverse components, these equations become:

$$\nabla_t\cdot\mathbf{E}_t + \frac{\partial E_z}{\partial z} = 0 \quad , \qquad \nabla_t\cdot\mathbf{B}_t + \frac{\partial B_z}{\partial z} = 0 \quad ,$$

$$\left(\nabla_t + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\times(\mathbf{E}_t + E_z\hat{\mathbf{z}}) = i\omega(\mathbf{B}_t + B_z\hat{\mathbf{z}}) \quad \text{, and}$$

$$\left(\nabla_t + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\times(\mathbf{B}_t + B_z\hat{\mathbf{z}}) = -i\omega\mu\epsilon(\mathbf{E}_t + E_z\hat{\mathbf{z}})$$

- Evaluate the partial derivative with respect to z based on our assumption above. Also distribute through the cross product in the last two equations. We find:

$\boxed{\nabla_t \cdot \mathbf{E}_t = -ikE_z}$, $\boxed{\nabla_t \cdot \mathbf{B}_t = -ikB_z}$,

$$\nabla_t \times \mathbf{E}_t + \nabla_t \times E_z \hat{\mathbf{z}} + ik\hat{\mathbf{z}} \times \mathbf{E}_t = i\omega \mathbf{B}_t + i\omega B_z \hat{\mathbf{z}}$$, and

$$\nabla_t \times \mathbf{B}_t + \nabla_t \times B_z \hat{\mathbf{z}} + ik\hat{\mathbf{z}} \times \mathbf{B}_t = -i\omega\mu\epsilon \mathbf{E}_t - i\omega\mu\epsilon E_z \hat{\mathbf{z}}$$

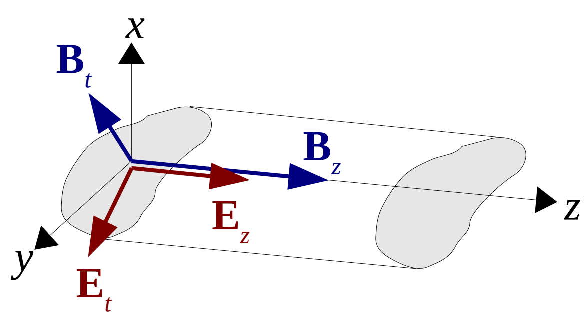
- The last two equations may look complicated at first, but we have done nothing fancy beyond assuming a harmonic time and z dependence and expanded out vector components.
- The last two equations above evaluate ultimately to a vector equaling a vector.
- We can further break down the resultant vector into parallel and transverse components.
- Dot the last two equations with $\hat{\mathbf{z}}$ to find the parallel components:

$\boxed{\hat{\mathbf{z}} \cdot (\nabla_t \times \mathbf{E}_t) = i\omega B_z}$ and $\boxed{\hat{\mathbf{z}} \cdot (\nabla_t \times \mathbf{B}_t) = -i\mu\epsilon\omega E_z}$

- The four equations in boxes above directly give us the axial components of the fields if we know the transverse components. Let's try to find equations that will do the opposite.
- Instead of dotting, now cross the two long equations above with $\hat{\mathbf{z}}$ to find the transverse components.

$$\hat{\mathbf{z}} \times \left[\nabla_t \times \mathbf{E}_t + \nabla_t \times E_z \hat{\mathbf{z}} + ik\hat{\mathbf{z}} \times \mathbf{E}_t\right] = \hat{\mathbf{z}} \times \left[i\omega \mathbf{B}_t + i\omega B_z \hat{\mathbf{z}}\right]$$ and

$$\hat{\mathbf{z}} \times \left[\nabla_t \times \mathbf{B}_t + \nabla_t \times B_z \hat{\mathbf{z}} + ik\hat{\mathbf{z}} \times \mathbf{B}_t\right] = \hat{\mathbf{z}} \times \left[-i\omega\mu\epsilon \mathbf{E}_t - i\omega\mu\epsilon E_z \hat{\mathbf{z}}\right]$$



- Many terms disappear because of the cross product of parallel vectors, giving:

$$\hat{\mathbf{z}} \times (\nabla_t \times E_z \hat{\mathbf{z}}) + ik\hat{\mathbf{z}} \times \hat{\mathbf{z}} \times \mathbf{E}_t = i\omega \hat{\mathbf{z}} \times \mathbf{B}_t$$ and $$\hat{\mathbf{z}} \times (\nabla_t \times B_z \hat{\mathbf{z}}) + ik\hat{\mathbf{z}} \times \hat{\mathbf{z}} \times \mathbf{B}_t = -i\omega\mu\epsilon \hat{\mathbf{z}} \times \mathbf{E}_t$$

- Now use the vector identities mentioned at the beginning to reduce these equations to:

$$\nabla_t E_z - ik\mathbf{E}_t = i\omega \hat{\mathbf{z}} \times \mathbf{B}_t$$ and $$\nabla_t B_z - ik\mathbf{B}_t = -i\omega\mu\epsilon \hat{\mathbf{z}} \times \mathbf{E}_t$$

- Solve for the transverse components by substituting back and forth to find:

$$\boxed{\mathbf{E}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\left(k\nabla_t E_z - \omega \hat{\mathbf{z}} \times \nabla_t B_z\right)}$$ and $$\boxed{\mathbf{B}_t = \frac{i}{\mu\epsilon\omega^2 - k^2}\left(k\nabla_t B_z + \mu\epsilon\omega \hat{\mathbf{z}} \times \nabla_t E_z\right)}$$

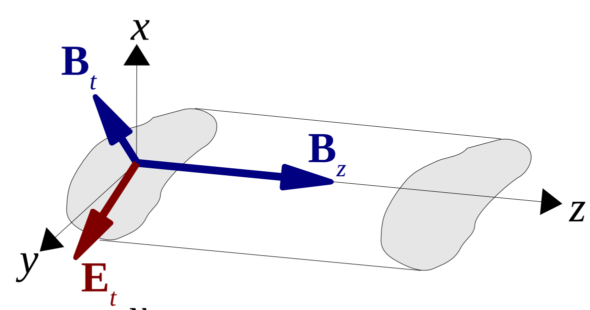
- These are known as the waveguide equations.

- If we know the axial components, we can plug them into these equations and directly solve for the transverse components.
- Note that we have not yet applied the effects of the walls of the waveguide. All we have done is assume that the fields in the *z* direction form single-frequency traveling waves. These equations are forced upon us by Maxwell's equations.
- Also note that these equations do not work for purely transverse waves (TEM waves). Setting $B_z = 0$ and $E_z = 0$ in these equations, and recognizing that $(\mu\varepsilon\omega^2 - k^2) = 0$ for TEM waves as in free space, we end up with $E_t = 0/0$ and $B_t = 0/0$. While these statements could be construed to be true mathematically, they are not useful.
- The different particular field patterns allowed in a waveguide are called "modes".
- We can break up the equations above into two cases and treat them as separate modes: one case where $E_z = 0$ and the other case where $B_z = 0$.
- If $E_z = 0$, this means that the electric field is purely transverse. We call this a "transverse electric" mode or "TE" mode.
- If $B_z = 0$, this means that the magnetic field is purely transverse. We call this a "transverse magnetic" mode or "TM" mode.
- Using these definitions, the waveguide equations reduce down to:

TRANSVERSE ELECTRIC (TE) WAVE:

$$\mathbf{E}_t = \frac{-i\omega}{\mu\epsilon\omega^2 - k^2}\left(\hat{\mathbf{z}} \times \nabla_t B_z\right)$$

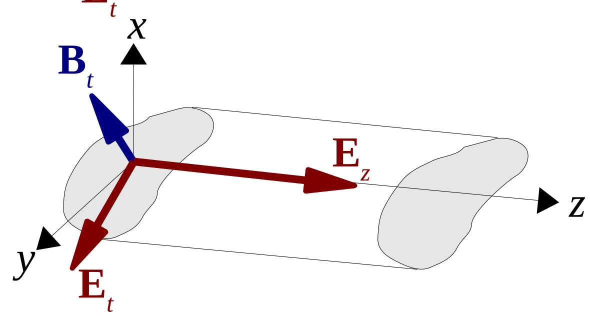
$$\mathbf{B}_t = \frac{ik}{\mu\epsilon\omega^2 - k^2}\left(\nabla_t B_z\right)$$



TRANSVERSE MAGNETIC (TM) WAVE:

$$\mathbf{E}_t = \frac{ik}{\mu\epsilon\omega^2 - k^2}\left(\nabla_t E_z\right)$$

$$\mathbf{B}_t = \frac{i\mu\epsilon\omega}{\mu\epsilon\omega^2 - k^2}\left(\hat{\mathbf{z}} \times \nabla_t E_z\right)$$

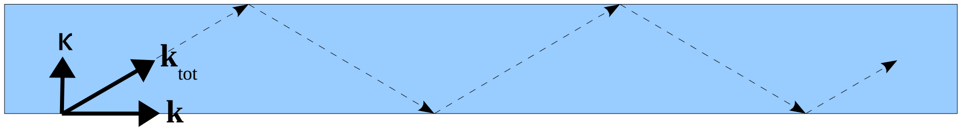


- If we treat the fields inside a waveguide not as complicated waves with axial components traveling directly down the waveguide axis, but instead as regular transverse waves traveling at an oblique angle and repeatedly bouncing off the walls and down the waveguide, then the free-space dispersion relation still holds:

$$k_{\text{tot}} = \sqrt{\epsilon\mu}\,\omega \quad \text{or} \quad k_{\text{tot}}^2 = \epsilon\mu\,\omega^2$$

- Here, $\mathbf{k}_{\text{tot}}$ is the total wavevector. It points at some oblique angle to the waveguide axis, specifying the direction that the transverse-like waves travel. It can be broken down into the axial wavevector component $\mathbf{k}$ and the transverse wavevector component $\boldsymbol{\kappa}$ (lowercase Greek letter kappa):

$$\mathbf{k}_{\text{tot}} = \mathbf{k} + \boldsymbol{\kappa}$$



- Using the Pythagorean theorem, we have:

$$k_{tot}^2 = k^2 + \kappa^2$$

- Apply this expansion to the down-waveguide dispersion relation:

$$k_{tot}^2 = \epsilon\mu\,\omega^2$$

$$k^2 + \kappa^2 = \epsilon\mu\,\omega^2$$

$$\boxed{k = \sqrt{\epsilon\mu\,\omega^2 - \kappa^2}}$$ *Waveguide Dispersion Relation*

- If the frequency is too low (if $\epsilon\mu\,\omega^2 < \kappa^2$), the argument under the square root will be negative. The square root of a negative number is imaginary, indicating that the wavenumber is purely imaginary-valued (assuming that the permittivity and permeability are real-valued). Therefore, we can only have traveling waves in the waveguide at high enough frequencies. At frequencies that are too low, the imaginary value of the wavenumber indicates that the fields form non-traveling, strongly-attenuated, standing waves (evanescent waves) inside the waveguide.
- We can rearrange the dispersion relation into the form:

$$\kappa^2 = \epsilon\mu\,\omega^2 - k^2$$

- The terms on the right in the equation above are what appear in the denominator of the waveguide equations, so we can simplify the waveguide equations to:

TE: $\boxed{\mathbf{E}_t = -i\,\frac{\omega}{\kappa^2}\left(\hat{\mathbf{z}}\times\nabla_t B_z\right)}$ $\boxed{\mathbf{B}_t = i\,\frac{k}{\kappa^2}\left(\nabla_t B_z\right)}$

TM: $\boxed{\mathbf{E}_t = i\,\frac{k}{\kappa^2}\left(\nabla_t E_z\right)}$ $\boxed{\mathbf{B}_t = i\,\frac{\mu\epsilon\omega}{\kappa^2}\left(\hat{\mathbf{z}}\times\nabla_t E_z\right)}$

- The transverse wavevector **κ** is a constant for a given mode since the fields can only form certain standing waves in the transverse direction. Once the boundary conditions on the waveguide walls are applied, the possible values of **κ** are known and form a discrete set. Since κ is known for a given mode, and ω is known (since it is an initially-known quantity that is controlled by whatever agent is creating the fields in the first place), the axial wavenumber k becomes known through the dispersion relation.