Lecture 3 Notes, Electromagnetic Theory II

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1. Superposition of Waves

- There is no such thing as an exactly monochromatic wave.
- A wave cannot have always been generated in the infinite past and continue to be generated in the infinite future, and therefore spread across an infinite span of space.
- Every physical electromagnetic wave therefore has a beginning point and an endpoint in space. An exactly monochromatic wave cannot exist because it would require an infinite extent and infinite existence.
- If you zoom out enough, every wave is actually a wave packet.
- Many frequency components are required in order to build up a wave packet, so every real wave always has a range of frequencies present, and not just one.
- Wave packets that contain a sine wave and have a very large spatial width contain only a narrow range of frequency components and may be approximated as monochromatic waves.
- For a nearly monochromatic wave, the width of the wave's range of frequencies is known as the "spectral linewidth".
- Because a dispersive material has a permittivity that depends on frequency, the different frequency components of a wave packet interact differently with the material.
- For simplicity, we will investigate the behavior of one vector component of the electric field and label it $u(x, t)$ so that for instance $E_x(x, t) = u(x, t)$. Because the components of a vector are independent, the end solution is just the sum of each vector component after being solved separately.
- As found previously, the principal solution of the wave equation is:

$$u(x,t)=A e^{i(kx-\omega t)} \quad \text{where} \quad k=\sqrt{\mu(\omega)\epsilon(\omega)}\,\omega$$

- We will treat the wavenumber k as the independent variable so that the frequency ω is a function of the wavenumber.
- Assume for now that ω and k are real.
- The general solution is just the superposition of all possible solutions:

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{i(kx-\omega(k)t)}\, dk$$

General Solution to the Wave Equation

- The coefficient function $A(k)$ in the general solution is what uniquely determines a specific waveform.
- The expression above is just a Fourier transform and we can apply Fourier theory to find the coefficients.
- Suppose we know the wave shape at some initial time $t = 0$. We apply this initial condition to find:

$$u(x,0)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)\, e^{ikx}\, dk$$

- Multiply both sides by a complex exponential and integrate both sides over *x*:

$$\int_{-\infty}^{\infty} u(x,0) e^{-ik'x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) \int_{-\infty}^{\infty} e^{i(k-k')x} dx\, dk$$

- Due to the orthogonality of complex exponentials, the inner integral on the right will be zero except when $k = k'$ at which it equals 2π. This collapses the outer integral allowing us to solve for $A(k)$:

$$\boxed{A(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,0) e^{-ikx} dx}$$ *Coefficients to the General Solution of the Wave Equation*

- The coefficient function $A(k)$ represents the wavenumber spectrum and its plot is referred to as "the wave in *k*-space".
- In non-dispersive materials, the wave number *k* and frequency ω are related by a constant so that the phrases "the wave in *k*-space" and "the wave in frequency space" can be used interchangeably.
- With a proper understanding of Fourier transforms, it now becomes evident that a wave with a large length in coordinate space corresponds to narrow wave-shape in *k*-space.
- The velocity at which a single-wavenumber component of the wave (typically the central wavenumber) travels is known as the "phase velocity".
- The velocity at which the wave packet moves as a whole is known as the "group velocity".
- The group velocity only has meaning insofar as the wave generally keeps the same shape over time.
- If there is low dispersion, we can expand the dispersion relation in a Taylor series and keep only the first two terms:

$$\omega(k) \approx \omega(k_0) + (k-k_0)\left[\frac{d\omega}{dk}\right]_{k=k_0}$$

- Substitute this into the general solution:

$$u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} A(k) e^{i\left(kx - \omega(k_0)t - (k-k_0)\left[\frac{d\omega}{dk}\right]_{k=k_0} t\right)} dk$$

$$u(x,t) = \frac{1}{\sqrt{2\pi}} e^{it\left(-\omega(k_0) + k_0\left[\frac{d\omega}{dk}\right]_{k=k_0}\right)} \int_{-\infty}^{\infty} A(k) e^{ik\left(x - \left[\frac{d\omega}{dk}\right]_{k=k_0} t\right)} dk$$

- Looking at the integral, we realize that is is just the initial waveform shifted forward by an amount $\left[\frac{d\omega}{dk}\right]_{k=k_0} t$, leading to:

$$u(x,t) = e^{i\theta_a t} u\left(x - \left[\frac{d\omega}{dk}\right]_{k=k_0} t, 0\right)$$

- Thus, except for a phase factor, the wave-shape stays the same and travels forward at the speed $\left[\frac{d\omega}{dk}\right]_{k=k_0}$ which we call the group velocity:

$$u(x,t)=e^{i\theta_a t}u\left(x-v_g t,0\right) \quad \text{where} \quad \boxed{v_g=\left[\frac{d\omega}{dk}\right]_{k=k_0}}$$

- It is worth repeating that the group velocity v_g was derived using an approximation which is only valid when the dispersion relation curve is smooth.
- In some situations, we can mathematically calculate the group velocity and find it to be greater than the speed of light in vacuum. This odd result only indicates that the approximation has become invalid due to a non-smooth dispersion relation, and therefore the group velocity has lost physical meaning. In such situations, the wave packet becomes so distorted by dispersion as it travels that we can not meaningfully assign a speed to the packet as a whole.
- We can use the basic dispersion relation, $k=\sqrt{\epsilon\mu}\,\omega$ (assume that the magnetic permeability is constant, but the electric permittivity is not), to find the group velocity in terms of the permittivity.
- Take the derivative of the dispersion relation on both sides with respect to *k*.

$$1=\sqrt{\epsilon\mu}\frac{d\omega}{dk}+\omega\sqrt{\mu}\frac{1}{2}\frac{1}{\sqrt{\epsilon}}\frac{d\epsilon}{d\omega}\frac{d\omega}{dk}$$

$$1=\sqrt{\epsilon\mu}\,v_g+\omega\sqrt{\mu}\frac{1}{2}\frac{1}{\sqrt{\epsilon}}\frac{d\epsilon}{d\omega}v_g$$

- Now solve for the group velocity:

$$v_g=\frac{1}{\sqrt{\epsilon\mu}+\frac{\omega\sqrt{\mu}}{2\sqrt{\epsilon}}\frac{d\epsilon}{d\omega}}$$

$$\boxed{v_g=\frac{c}{\sqrt{\epsilon_r\mu_r}+\frac{\omega\sqrt{\mu_r}}{2\sqrt{\epsilon_r}}\frac{d\epsilon_r}{d\omega}}}$$

- In contrast, the phase velocity is defined as:

$$v_p=\frac{\omega}{k}$$

$$\boxed{v_p=\frac{1}{\sqrt{\epsilon\mu}}} \quad \text{or} \quad \boxed{v_p=\frac{c}{\sqrt{\epsilon_r\mu_r}}} \quad \text{or} \quad \boxed{v_p=\frac{c}{n}}$$

- It should be obvious that when there is no dispersion (the permittivity does not depend on frequency), the group velocity equals the phase velocity.

2. Illustrating Pulse Spreading in a Dispersive Medium

- Consider a wave packet which is a cosine inside a Gaussian envelope at $t = 0$:

$$u(x,0)=e^{-x^2/2L^2}\cos(k_0 x)$$

where L is the width of the wave packet.
- As the wave propagates through a medium, its form at any later time t is given by:

$$u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A(k)e^{i(kx-\omega(k)t)}dk \quad \text{where} \quad A(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)e^{-ikx}dx$$

$$A(k)=\frac{L}{2}\left[e^{-(L^2/2)(k-k_0)^2}+e^{-(L^2/2)(k+k_0)^2}\right]$$

$$u(x,t)=\frac{L}{2}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[e^{-(L^2/2)(k-k_0)^2}+e^{-(L^2/2)(k+k_0)^2}\right]e^{i(kx-\omega(k)t)}dk$$

- We must know $\omega(k)$ before we can do the last integral. For the purpose of the illustration, assume the dispersion relation is:

$$\omega(k)=\omega_0\left(1+\frac{1}{2}a^2k^2\right)$$

- Plugging this in and doing the integral gives a solution of the form:

$$u(x,t)=u_0e^{ig(x,t)}\exp\left[\frac{-(x-\omega_0 a^2 k_0 t)^2}{2L^2(t)}\right] \quad \text{where} \quad L(t)=\sqrt{L^2+(a^2\omega_0 t/L)^2}$$

- The exact form of the amplitude of the final solution as well as the oscillating part are unimportant to this illustration and have been tucked inside u_0 and $g(x, t)$.
- What is important to note is that the envelope of the wave (the last factor) is still a Gaussian, but it is a Gaussian that has been shifted in space because it is traveling at a velocity $v_g = \omega_0 a^2 k_0$.
- Also of note is that the width of the envelope is increasing with time, and is dependent on several factors: materials with higher dispersion factors a and ω_0 cause the wave to spread out faster. Also, a narrower initial wave pulse spreads out faster.

3. Causality

- When the permittivity becomes dependent on frequency, there becomes a connection between **D** and **E** that is non-local in time.
- For a monochromatic component:

$$\mathbf{D}(\mathbf{x},\omega)=\epsilon(\omega)\mathbf{E}(\mathbf{x},\omega)$$

- Fourier transform both sides to deduce the temporal relationship:

$$\mathbf{D}(\mathbf{x},t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\epsilon(\omega)\mathbf{E}(\mathbf{x},\omega)e^{-i\omega t}d\omega$$

- Fourier transform the electric field **E** as well.

$$\mathbf{D}(\mathbf{x},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\epsilon(\omega)\int_{-\infty}^{\infty}\mathbf{E}(\mathbf{x},t')e^{i\omega t'}dt'e^{-i\omega t}d\omega$$

$$\mathbf{D}(\mathbf{x},t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}d\omega\,\epsilon(\omega)e^{i\omega(t'-t)}\mathbf{E}(\mathbf{x},t')$$

- Add and subtract the electric field **E** to get **D** in a form where we can identify the polarization:

$$\mathbf{D}(\mathbf{x},t)=\epsilon_0\mathbf{E}(\mathbf{x},t)-\epsilon_0\mathbf{E}(\mathbf{x},t)+\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}d\omega\,\epsilon(\omega)e^{i\omega(t'-t)}\mathbf{E}(\mathbf{x},t')$$

$$\mathbf{D}(\mathbf{x},t)=\epsilon_0\mathbf{E}(\mathbf{x},t)-\epsilon_0\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}d\omega\,e^{i\omega(t'-t)}\mathbf{E}(\mathbf{x},t')+\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}d\omega\,\epsilon(\omega)e^{i\omega(t'-t)}\mathbf{E}(\mathbf{x},t')$$

$$\mathbf{D}(\mathbf{x},t)=\epsilon_0\mathbf{E}(\mathbf{x},t)+\epsilon_0\frac{1}{2\pi}\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}d\omega\,e^{i\omega(t'-t)}\mathbf{E}(\mathbf{x},t')\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]$$

- Make a change of integration variables: $t'\rightarrow t-\tau$

$$\mathbf{D}(\mathbf{x},t)=\epsilon_0\mathbf{E}(\mathbf{x},t)+\epsilon_0\frac{1}{2\pi}\int_{-\infty}^{\infty}d\tau\int_{-\infty}^{\infty}d\omega\,e^{-i\omega\tau}\mathbf{E}(\mathbf{x},t-\tau)\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]$$

$$\mathbf{D}(\mathbf{x},t)=\epsilon_0\mathbf{E}(\mathbf{x},t)+\epsilon_0\int_{-\infty}^{\infty}G(\tau)\mathbf{E}(\mathbf{x},t-\tau)d\tau \quad \text{where} \quad G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,e^{-i\omega\tau}\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]$$

- Only electric fields at some prior time τ can effect the field at the current time t, so we can make the limit on the lower integration zero.

$$\boxed{\mathbf{D}(\mathbf{x},t)=\epsilon_0\mathbf{E}(\mathbf{x},t)+\epsilon_0\int_{0}^{\infty}G(\tau)\mathbf{E}(\mathbf{x},t-\tau)d\tau} \quad \text{where} \quad \boxed{G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\,e^{-i\omega\tau}\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]}$$

- The variable $G(\tau)$ is the Fourier transform of the susceptibility.
- The second term on the right of the first equation is the polarization **P**.
- We can check this result. For non-dispersive materials, the permittivity does not depend on frequency and can come out of the integral, so that G reduces to:

$$G(\tau)=\left(\frac{\epsilon}{\epsilon_0}-1\right)\delta(\tau)$$

- Plugging this into the causality relationship gives:

$$\mathbf{D}(\mathbf{x},t)=\epsilon_0\mathbf{E}(\mathbf{x},t)+\epsilon_0\left(\frac{\epsilon}{\epsilon_0}-1\right)\mathbf{E}(\mathbf{x},t)$$

$$\mathbf{D}(\mathbf{x},t)=\epsilon\,\mathbf{E}(\mathbf{x},t)$$

- This is what we expect for non-dispersive media.
- For dispersive media, the causality relationship tells us that polarization depends on the electric field at different points of time weighted by the susceptibility of the material at that point in time.
- Let us invert the kernel equation:

$$G(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega e^{-i\omega\tau}\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]$$

$$\int_0^{\infty}G(\tau)e^{i\omega'\tau}d\tau=\frac{1}{2\pi}\int_0^{\infty}d\tau e^{i\omega'\tau}\int_{-\infty}^{\infty}d\omega e^{-i\omega\tau}\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]$$

$$\int_0^{\infty}G(\tau)e^{i\omega'\tau}d\tau'=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]\int_0^{\infty}d\tau e^{i(\omega'-\omega)\tau}$$

$$\int_0^{\infty}G(\tau)e^{i\omega'\tau}d\tau'=\int_{-\infty}^{\infty}d\omega\left[\frac{\epsilon(\omega)}{\epsilon_0}-1\right]\delta(\omega'-\omega)$$

$$\boxed{\frac{\epsilon(\omega)}{\epsilon_0}=1+\int_0^{\infty}G(\tau)e^{i\omega\tau}d\tau'}$$

- Let us assume for the moment that the frequency could be some complex number *z*.
- This relation can be used to show that the permittivity is analytic, and Cauchy's Theorem can thus be used.
- Cauchy's theorem states that a closed line integral in the complex plane of some function with a singular point *z* is proportional to the value of the function at that point:

$$\oint\frac{f(\omega')}{\omega'-z}d\omega'=2\pi i\,f(z)\qquad\textit{Cauchy's Theorem}$$

- For our case here, the susceptibility is analytic and becomes the function: $f=\chi_e=\varepsilon(\omega)/\varepsilon_0-1$

$$\oint\frac{\epsilon(\omega')/\epsilon_0-1}{\omega'-z}d\omega'=2\pi i(\epsilon(z)/\epsilon_0-1)$$

$$\epsilon(z)/\epsilon_0=1+\frac{1}{2\pi i}\oint\frac{\epsilon(\omega')/\epsilon_0-1}{\omega'-z}d\omega'$$

- To match reality, we align the contour integral so that it sweeps the real axis and then follows a semi-circle at infinity. The semi-circle piece can be dropped. We must be careful to go just around the pole:

$$\epsilon(\omega)/\epsilon_0 = 1 + \frac{1}{2}(\epsilon(\omega)/\epsilon_0 - 1) + \frac{1}{2\pi i} P \int_{-\infty}^{\infty} \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - \omega} d\omega'$$

$$\epsilon(\omega)/\epsilon_0 = 1 + \frac{1}{\pi i} P \int_{-\infty}^{\infty} \frac{\epsilon(\omega')/\epsilon_0 - 1}{\omega' - \omega} d\omega'$$

- The operator *P* means take the principal part, or in other words, do the integral over all points except the singular points.
- If we split this equation into its real and imaginary components, we get:

$$\boxed{\Re(\epsilon(\omega)/\epsilon_0) = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Im(\epsilon(\omega')/\epsilon_0)}{\omega' - \omega} d\omega'}$$

$$\boxed{\Im(\epsilon(\omega)/\epsilon_0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\Re(\epsilon(\omega')/\epsilon_0) - 1}{\omega' - \omega} d\omega'}$$

- These are the Kramers-Kronig Relations. They are very useful experimentally. Typically, one measures the imaginary part of the permittivity through absorption studies and can then calculate directly the real part.