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## Lecture 2 Notes, Electromagnetic Theory II

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### 1. Dispersion Introduction

- An electromagnetic wave with an arbitrary wave-shape can be thought of as a superposition of single-frequency plane waves.
- If the dielectric material responds in the exact same way to plane waves of any frequency, then each component of the wave-shape will travel at the same speed, and the overall wave-shape will be preserved.
- If the dielectric material responds differently to plane waves of different frequencies (i.e. the material's dielectric constant is frequency dependent), then the various components of a wave-shape will travel at different speeds. The wave-shape will change in time as it propagates through the material and in general will spread out.
- This wave spreading is known as dispersion.
- Before we can describe dispersion mathematically, we must understand the behavior of the frequency-dependent permittivity  $\epsilon(\omega)$ .
- An accurate prediction of a material's permittivity requires quantum mechanics, but we can find a good approximation by building a classical model.

### 2. Classical Harmonic Model

- We want to develop a simple model that describes the frequency dependence of a material's permittivity.
- Remember that the electric permittivity is the material's response to an applied electric field:

$$\epsilon = \frac{D}{E}$$

or in terms of the polarization:

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{1}{\epsilon_0} \frac{P}{E}$$

- The applied field induces dipoles in the material which then create their own electric fields that add to the original field.
- The polarization  $\mathbf{P}$  is the average induced electric dipole moment density.
- Assuming that a total number  $n$  of individual induced dipoles, each of strength  $p$ , are distributed uniformly through a volume  $V$ , the polarization is:

$$P = \frac{n}{V} p$$

- Therefore the permittivity is:

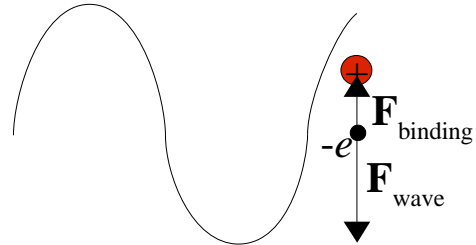
$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{n}{\epsilon_0 V} \frac{p}{E}$$

- If we find the dipole moment  $p$  induced in a single atom by the field of a single-frequency wave passing by, we will know the permittivity of the material as a function of frequency.
- For simplicity, we will deal only with non-magnetic dielectric materials, so that the magnetic permeability equals the permeability of free space. We will also ignore magnetic forces since the magnetic force from a plane wave is typically much smaller than the electric force.
- Consider a single electron that is bound to an atomic nucleus and is pushed by a monochromatic electromagnetic wave that passes by. Assume the atomic nucleus is so much heavier than the electron that it stays fixed.
- We will treat the atomic binding force as harmonic, so that the electron acts like a mass on a spring.
- Newton's Law states:

$$\mathbf{F} = m \mathbf{a}$$

where  $\mathbf{F}$  is the total force,  $m$  is the electron's mass and  $\mathbf{a}$  is the electron's acceleration.

- There are three forces: the force from the electromagnetic wave passing by that tries to displace the electron, the binding force of the atom that tries to restore the electron to its equilibrium point, and a damping force that is proportional to the electron's velocity and tends to oppose its motion.



$$\mathbf{F}_{\text{wave}} + \mathbf{F}_{\text{binding}} + \mathbf{F}_{\text{damping}} = m \frac{d^2 \mathbf{x}}{dt^2}$$

$$-e \mathbf{E} - k_s \mathbf{x} - m \gamma \frac{d \mathbf{x}}{dt} = m \frac{d^2 \mathbf{x}}{dt^2}$$

- where  $k_s$  is the spring constant of the harmonic binding force and  $\gamma$  is the damping parameter.
- For a harmonic system with a single mass the spring constant is  $k_s = m \omega_0^2$  where  $\omega_0$  is the resonant frequency of the spring (which is here caused by the binding force in the atom).

$$-e \mathbf{E} - m \omega_0^2 \mathbf{x} - m \gamma \frac{d \mathbf{x}}{dt} = m \frac{d^2 \mathbf{x}}{dt^2}$$

$$\frac{e}{m} \mathbf{E} + \omega_0^2 \mathbf{x} + \gamma \frac{d \mathbf{x}}{dt} + \frac{d^2 \mathbf{x}}{dt^2} = 0$$

- Let us investigate the material's response to a plane wave at a single frequency  $\omega$ , and then we can always super-impose plane waves for more complex situations:

$$\frac{e}{m} \mathbf{E}(\mathbf{x}) e^{-i\omega t} + \omega_0^2 \mathbf{x} + \gamma \frac{d \mathbf{x}}{dt} + \frac{d^2 \mathbf{x}}{dt^2} = 0$$

- We try a solution of the form  $\mathbf{x} = \mathbf{x}_0 e^{-i\omega t}$ :

$$\frac{e}{m} \mathbf{E}(\mathbf{x}) e^{-i\omega t} + \omega_0^2 \mathbf{x}_0 e^{-i\omega t} - i\omega \gamma \mathbf{x}_0 e^{-i\omega t} - \omega^2 \mathbf{x}_0 e^{-i\omega t} = 0$$

$$\frac{e}{m} \mathbf{E} + \omega_0^2 \mathbf{x} - i\omega \gamma \mathbf{x} - \omega^2 \mathbf{x} = 0$$

$$\mathbf{x} = -\frac{e}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega \gamma)} \mathbf{E}$$

- The induced dipole moment of this simplified atom then becomes:

$$\mathbf{p} = -e \mathbf{x}$$

$$\mathbf{p} = \frac{e^2}{m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega \gamma)} \mathbf{E}$$

*Dipole moment of a single electron within the harmonic model*

- From this equation we see that the induced dipole moment in an atom has its direction aligned with the electric field of the wave, the dipole oscillates at the same frequency as the wave, and the dipole lags in phase by some amount determined by the phase of the denominator.

- This dipole moment expression is plugged into our definition of the permittivity to find:

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{n e^2}{\epsilon_0 V m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega \gamma)}$$

- Defining the atomic number density  $N$  according to  $N = n/V$ , this expression becomes:

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{N e^2}{\epsilon_0 m} \frac{1}{(\omega_0^2 - \omega^2 - i\omega \gamma)}$$

- If instead of one electron per atom we have  $f$  electrons per atom in the same binding state, then we must multiply the original polarization by  $f$  to get the overall polarization, leading to:

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{N e^2}{\epsilon_0 m} \frac{f}{(\omega_0^2 - \omega^2 - i\omega \gamma)}$$

- To be even more general, we should consider that there are different electrons in different bound states in a single atom, each contributing to the material's response:

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{N e^2}{\epsilon_0 m} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\omega \gamma_j} \quad \text{Dielectric Constant in the harmonic model, non-conducting}$$

- The resonant frequencies  $\omega_j$ , the oscillator strengths  $f_j$ , and the damping coefficients  $\gamma_j$  must be predicted by further models or found experimentally.

- We can tuck away a lot of the constants into a single constant called the plasma frequency:

$$\frac{\epsilon}{\epsilon_0} = 1 + \omega_p^2 \sum_j \frac{f_j / f_0}{\omega_j^2 - \omega^2 - i \omega \gamma_j} \quad \text{where} \quad \omega_p = \sqrt{\frac{N f_0 e^2}{\epsilon_0 m}}$$

- The plasma frequency is the frequency at which a charge displacement in an ideal plasma will naturally oscillate if left to itself. It is also the threshold frequency for an ideal plasma. Below this threshold incident waves are strongly reflected and above it incident waves are strongly transmitted.
- The parameter  $f_0$  is the oscillator strength of the highest resonance. It becomes the number of free electrons per atom (also labeled  $Z$ ) if we are dealing with an ideal plasma.
- Even though the plasma frequency has a precise physical meaning only for an ideal plasma, we can still use it for other materials as a way to tuck away all those constants. Furthermore, when materials (such as metals) act approximately like an ideal plasma, the plasma frequency can still have its usual physical meaning in an approximate way.

### 3. Plotting the Dielectric Constant

- At very small frequencies,  $\omega \ll \omega_j$ , the equation reduces to:

$$\frac{\epsilon}{\epsilon_0} = \epsilon_s = 1 + \sum_j \frac{\omega_p^2}{\omega_j^2} \frac{f_j}{f_0} \quad \text{Static Dielectric Constant}$$

- This is known as the static dielectric constant because it is independent of frequency, and is thus valid for static fields as well as low-frequency waves. Typically, whenever an author chooses to ignore the frequency behavior of a material, he or she is using the static dielectric constant.
- Note that the expression above is real-valued, therefore there is no imaginary part to the dielectric constant at low frequencies.
- For the purpose of plotting, the real part of the dielectric constant curve should smoothly approach a constant value at low frequencies, and the imaginary part should approach zero.
- On the other extreme, at very high frequencies,  $\omega \gg \omega_j$ , the general equation reduces to:

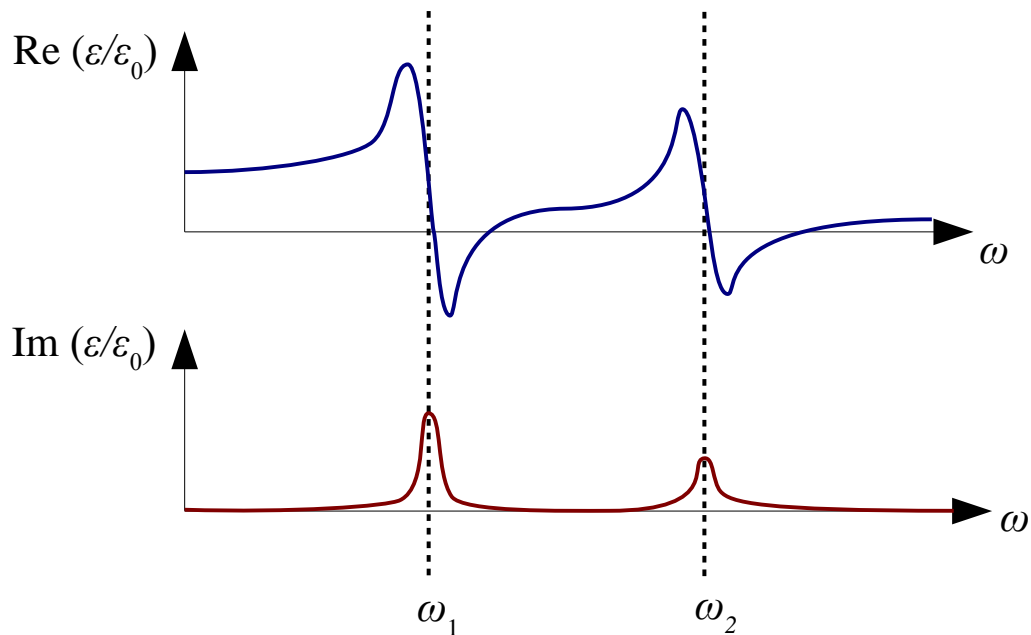
$$\frac{\epsilon}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2}$$

- This is also the expression for an ideal plasma, which will be discussed in a later section.
- Therefore, at high enough frequencies, all materials act somewhat like an ideal plasma.
- Note that this expression is also real-valued.
- For the purpose of plotting, the real part of the permittivity at high frequencies should start negative and gradually slope up to the value of 1, while the imaginary part should approach zero.
- Very close to a resonant frequency,  $\omega \approx \omega_j$ , the quantity  $(\omega_j^2 - \omega^2)$  becomes approximately zero and the damping term takes over:

$$\frac{\epsilon}{\epsilon_0} = 1 + i \frac{\omega_p^2}{\omega_j \gamma_j} \frac{f_j}{f_0} \quad \text{Near the } j^{\text{th}} \text{ resonance}$$

- This is a constant number containing a real part of 1 and a positive imaginary part which depends on the constants of the material.

- We can put together all of these concepts and create a sketch:



- Dispersion (wave-spreading) is caused by different frequency components of a wave packet experiencing different different permittivities and thus traveling at different speeds. Therefore, dispersion is strong at frequencies where the blue curve above has a large slope.
- In frequency regions where the real part of the dielectric constant is flat, there is essentially no dispersion.
- There is greater dispersion closer to resonant frequencies.
- Once the driving frequency of the incoming wave is greater than a particular resonant frequency, the corresponding electron in the atom cannot keep up with the wave. Its motion gets out of phase with the wave and the dipole moment starts to oppose the field instead of align with it. That is why the permittivity dips so drastically.
- At a wave frequency much higher than a particular resonant frequency, the corresponding electron is too slow to respond at all and makes no contribution to the dielectric constant.
- In general, the real part of the dielectric constant of any material decreases and approaches that of vacuum for higher and higher frequencies as fewer and fewer electrons can respond.
- For most wave packets, higher frequency components experience a higher permittivity and thus move at a slower velocity. This is known as normal dispersion.
- For the frequency ranges directly near the resonant frequencies, the slope is negative. This is known as anomalous dispersion. The group velocity goes faster than the phase velocity for anomalous dispersion.

#### **4. Electric Conductivity**

- Our harmonic model assumed that all of the electrons in the atom (or molecule) are bound.
- What if the material also has some electrons that are not bound, or in other words, what if the material has a non-zero conductivity?
- We can still use the harmonic model if we realize that a free electron is just the limiting case of an weakly bound electron, where the binding force approaches zero.

- The weaker the binding force, the lower the resonant frequency.
- A free electron can thus be thought of as an bound electron that takes an infinite amount of time to return to the equilibrium point in its atom, and therefore has a resonant frequency of zero.
- We add the effects of the free electron by using the same model except with  $\omega_j = 0$  and  $f_j = f_0$ :

$$\frac{\epsilon}{\epsilon_0} = 1 + \omega_p^2 \sum_j \frac{f_j / f_0}{\omega_j^2 - \omega^2 - i \omega \gamma_j} + \frac{\omega_p^2}{-\omega^2 - i \omega \gamma_0}$$

$$\boxed{\frac{\epsilon}{\epsilon_0} = 1 + \omega_p^2 \sum_j \frac{f_j / f_0}{\omega_j^2 - \omega^2 - i \omega \gamma_j} + i \frac{\omega_p^2}{\omega(\gamma_0 - i \omega)}} \quad \text{where } \omega_p = \sqrt{\frac{N f_0 e^2}{\epsilon_0 m}} \quad \text{General Form}$$

- Here,  $f_0$  is now the number of free electrons per atom.
- The first two terms are just what we would have if there were no free electrons, so we can identify it as the bound dielectric constant  $\epsilon_b / \epsilon_0$ .

$$\boxed{\frac{\epsilon}{\epsilon_0} = \frac{\epsilon_b}{\epsilon_0} + i \frac{\omega_p^2}{\omega(\gamma_0 - i \omega)}} \quad \text{General Form}$$

- How does this relate to the conductivity  $\sigma$ ? We can answer this question by approaching Maxwell's equations in two ways: by assuming the free electrons are separate from the permittivity and are governed by Ohm's law, or by assuming the free electrons are very weakly bound electrons included in the permittivity.
- The Maxwell-Ampere equation is:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}$$

- Treat the free electrons as truly free, use Ohm's law  $\mathbf{J} = \sigma \mathbf{E}$  where  $\sigma$  is the conductivity, and also use  $\mathbf{D} = \epsilon_b \mathbf{E}$ :

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon_b \frac{d\mathbf{E}}{dt}$$

- We have harmonic time dependence so that:

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} - i \omega \epsilon_b \mathbf{E}$$

$$\nabla \times \mathbf{H} = -i \omega \epsilon_0 \left( \frac{\epsilon_b}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega} \right) \mathbf{E}$$

- If we instead treat the free electrons as very weakly bound electrons so that  $\mathbf{J} = 0$ , the Maxwell-Ampere equation with harmonic time dependence and  $\mathbf{D} = \epsilon \mathbf{E}$  becomes:

$$\nabla \times \mathbf{H} = -i \omega \epsilon_0 \left( \frac{\epsilon}{\epsilon_0} \right) \mathbf{E}$$

- Comparing the results of the two different approaches (the two equations above), we can make the identification:

$$\frac{\epsilon}{\epsilon_0} = \frac{\epsilon_b}{\epsilon_0} + i \frac{\sigma}{\epsilon_0 \omega}$$

*Dielectric Constant General Form*

- Comparing this with the results from our model lets us now define the conductivity according to our model:

$$\sigma = \frac{\epsilon_0 \omega_p^2}{\gamma_0 - i \omega}$$

*Conductivity According to the Harmonic Model*

## 5. Ideal Plasmas

- What if there were a material with only free electrons and the electrons are far enough apart that the damping is negligible? What would the dielectric constant look like?
- We start with the general form for our model:

$$\frac{\epsilon}{\epsilon_0} = 1 + \omega_p^2 \sum_j \frac{f_j / f_0}{\omega_j^2 - \omega^2 - i \omega \gamma_j} + i \frac{\omega_p^2}{\omega (\gamma_0 - i \omega)}$$

- We drop the second term which is for the bound electrons, and we set  $\gamma_0 = 0$  since damping is negligible, leading to:

$$\frac{\epsilon}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \quad \text{where} \quad \omega_p = \sqrt{\frac{N f_0 e^2}{\epsilon_0 m}}$$

*Dielectric Constant for Plasmas*

- When a material dominantly has free charges that are spaced far apart, it is a plasma.
- For an ideal plasma, this expression is valid at all frequencies. For all other materials, this expression is only approximately valid and only at high frequencies.
- Examples of plasmas include the ionosphere and electrically-charged gases in the laboratory.
- It is worth noting that there is no imaginary part to the dielectric constant for plasmas. This means that above the plasma frequency, the dielectric constant of a plasma is always positive and real-valued, indicating that the wavenumber  $k$  is purely real-valued. There is no loss and the wave passes through unimpeded. Below the plasma frequency, the dielectric constant is always negative and real-valued, indicating that the wavenumber  $k$  is purely imaginary. A wave incident on a large plasma below the plasma frequency will be completely reflected and the fields will penetrate only slightly into the plasma, dieing off exponentially.
- This is why radio waves are reflected by the ionosphere and are guided around the world for low-bandwidth world-wide communications, but higher-frequency microwaves pass through the ionosphere and must be relayed around the world using satellites for high-bandwidth communications.
- In a similar way, below the plasma frequency, highly doped semiconductors become reflective

and can be used as mirrors or waveguides. Doping semiconductors more strongly creates a higher free charge density, which leads to a higher plasma frequency and makes the material reflective to more frequencies. This has important implications in using doped semiconductors as custom-designed mirrors or filters.

- It should be remembered that at high enough frequencies, all materials behave like plasmas and waves pass through with little loss.
- Because the plasma frequency, which is the reflectivity threshold, depends directly on the charge density, we can use it to probe charge density. For instance, the atmosphere has a charge density that depends on altitude. One can map out the atmospheric charge density profile by sending a sweep of frequencies into the sky and determining which frequencies get reflected back from which point.

## **6. Wave Attenuation**

- Up until this lecture, we had only considered real-valued dielectric constants. This was not too bad of an oversight because the imaginary part of the dielectric constant is only non-zero near a resonant frequency.
- What does the addition of an imaginary part to the dielectric constant mean physically?
- Let us look first at what the imaginary part of the wave number  $k$  means.
- As found in last lecture, the general solution for a plane wave traveling in the  $x$  direction in a non-magnetic linear dielectric material is:

$$\mathbf{E} = \mathbf{E}_0 e^{i(kx - \omega t)} \quad \text{where} \quad k = \sqrt{\epsilon(\omega) \mu_0} \omega$$

- Because the permittivity is complex, the wavenumber must also be complex. Let us express this explicitly, defining the real part of the wavenumber as  $\beta$  and the imaginary part as  $(\alpha/2)$  so that  $k = \beta + i \alpha/2$ .

$$\mathbf{E} = \mathbf{E}_0 e^{i([\beta x + i \alpha]x - \omega t)}$$

$$\mathbf{E} = \mathbf{E}_0 e^{-(\alpha/2)x} e^{i(\beta x - \omega t)}$$

- It becomes obvious now that the real part of the wavenumber serves its usual role, setting the spatial scale of the oscillation, but the imaginary part of the wavenumber describes the exponential attenuation of the wave in space. In other words, the electric field strength gets weaker in the  $x$  direction.
- The wave can be attenuated by two things: by losing its energy to the material, or by losing its energy backwards as a reflected wave. In other words, since the wavenumber  $k$  is the square root of the permittivity,  $k$  can have an imaginary part in two ways: through the permittivity having an imaginary part (absorption), or through the permittivity being negative (reflection).
- If the material is thick enough, an attenuated wave will be completely absorbed or completely reflected so that no wave makes it out the other side of the material.
- The intensity of the wave, which is what is usually measured, becomes:

$$\langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\epsilon}{\mu}} E_0^2 e^{-\alpha x}$$

- This makes it obvious why the imaginary part of the wave number was defined with a factor of one half. The parameter  $\alpha$  is known as the attenuation coefficient. In contexts where there is



little reflection, it is also called the absorption coefficient.

- A material which has a significant imaginary part to its permittivity in a certain frequency range is known as a lossy material for these frequencies, since wave energy is lost to the heating up of the material.

- Let us explicitly connect the complex wave number and the complex permittivity using the relationship found originally:

$$k = \sqrt{\epsilon(\omega)} \mu_0 \omega$$

$$\beta + i\alpha/2 = \sqrt{\epsilon' + i\epsilon''} \sqrt{\mu_0} \omega \quad \text{where } \epsilon' \text{ denotes the real part of } \epsilon, \text{ and } \epsilon'' \text{ is the imaginary part}$$

$$\beta^2 - \alpha^2/4 + 2i\beta\alpha/2 = (\epsilon' + i\epsilon'') \mu_0 \omega^2$$

$$\boxed{\epsilon' = \frac{\beta^2 - \alpha^2/4}{\mu_0 \omega^2}} \quad \text{and} \quad \boxed{\epsilon'' = \frac{\beta\alpha}{\mu_0 \omega^2}}$$

*Complex Permittivity in Terms of the Wavenumber*

- These equations are used experimentally. The complex wavenumber is measured and then these equations are used to calculate the material's complex permittivity.

- We can invert these equations to understand the physics:

$$\boxed{\alpha = 2 \frac{\omega \sqrt{\mu_0}}{\sqrt{2}} \sqrt{|\epsilon| - \epsilon'}}$$

*Complex Wave Number in Terms of the Complex Permittivity*

$$\boxed{\beta = \frac{\omega \sqrt{\mu_0}}{\sqrt{2}} \sqrt{|\epsilon| + \epsilon'}}$$

- The meaning of all of these parameters can be summarized:

$\beta = \Re(k)$  : Refraction. (Higher  $\beta$  represents slower waves that bend more at interfaces.)

$\alpha = 2 \Im(k)$  : Attenuation. (Higher  $\alpha$  represents waves that die down quicker through space.)

$\epsilon' = \Re(\epsilon)$  : Energy Storage/Reemission. (Higher  $\epsilon'$  means energy is stored more strongly.)

$\epsilon'' = \Im(\epsilon)$  : Energy Loss. (Higher  $\epsilon''$  means more energy is converted to heat.)

- The equations in boxes above should make it clear that there is not a linear relationship between the real part of the permittivity and the real part of the wavenumber. In other words, energy storage does not automatically equal refraction. In the same way, there is not a linear relationship between the imaginary part of the permittivity and the imaginary part of the wavenumber. As a result, the material loss of a wave is not the exact same thing as the wave's attenuation. Let us look at a special case to make this clear:

- For a non-lossy material,  $\epsilon'' = 0$ , the equations reduce down to:

$$\text{If } \epsilon' > 0 \text{ then: } \beta = \omega \sqrt{\mu_0 \epsilon'} \quad \text{and} \quad \alpha = 0$$

This is an unattenuated traveling wave.

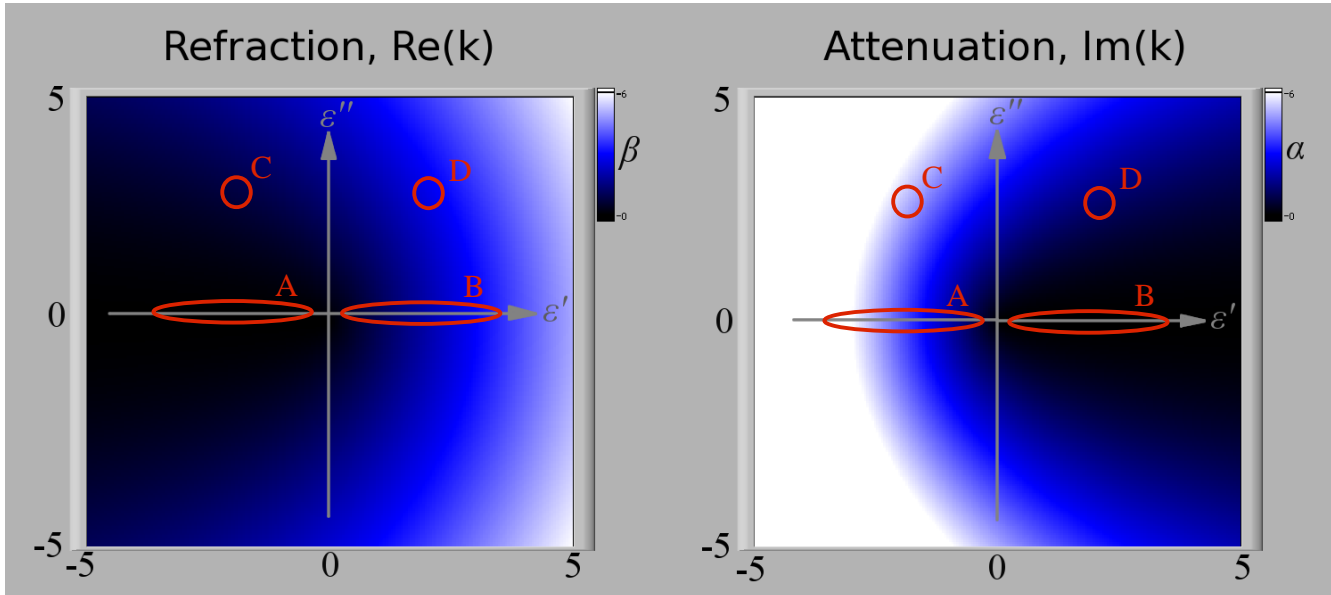
$$\text{If } \epsilon' < 0 \text{ then: } \beta = 0 \quad \text{and} \quad \alpha = 2\omega \sqrt{\mu_0 |\epsilon'|}$$

This is an attenuated standing wave.

- We see that waves can be attenuated even in a non-lossy material, if the waves are non-

traveling. This corresponds to an evanescent wave, where all of the wave energy is reflected back.

- We can plot the boxed equations above to understand what they mean:



- The  $x$  axis is the real part of the permittivity, the  $y$  axis is the imaginary part of the permittivity, and the color is the wavenumber. The real part of the wavenumber  $\beta$  is plotted on the left and the imaginary part of the wavenumber  $\alpha/2$  is plotted on the right. Black color indicates a value of zero and brighter colors indicate higher values.

- Points within circle A correspond to waves with infinite wavelength and strong attenuation. There are decaying standing waves, such as in plasmas below the plasma frequency. Waves incident on such a material would be strongly reflected. Note that there is no absorption because the imaginary part of the permittivity is zero, and yet we can still have a decaying wave shape.

- Points within circle B are non-attenuated traveling waves. There is no absorption by the material. Note that for everywhere except points in B, we get a non-zero  $\alpha$ , and therefore an attenuated waveshape. We see that both material absorption (non-zero  $\epsilon'' \neq 0$ ) and out-of-phase electrons (negative  $\epsilon' \neq 0$ ) can create a non-zero  $\alpha$ , and therefore a decaying waveshape.

- Points within circle C are strongly decaying, large-wavelength traveling waves.

- Points within circle D are slowly decaying, short-wavelength traveling waves.

- The best way to get strong refraction (the white region on the left image) is to have a strongly positive real part of the permittivity as well as a high imaginary part.

- The best way to get strong attenuation (the white region on the right image) is to have a strongly negative real part of the permittivity as well as a high imaginary part.