Lecture 13 Notes, Electromagnetic Theory II

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1. Covariant Geometry

- We would like to develop a mathematical framework in which Special Relativity can be applied more naturally.
- The Lorentz transformations were derived from Einstein's principle of relativity:

$$c^2 t'^2 - (x'^2 + y'^2 + z'^2) = c^2 t^2 - (x^2 + y^2 + z^2)$$

- This means that all the terms on the left always equal the same scalar no matter what frame of reference we are in. This value is invariant under Lorentz transformations.
- In regular three-dimensional Galilean relativity, the dot product of two position vectors is invariant under transformations.

Define the 4-vector (covariant) geometry as the set of rules that lead to the dot product of any two 4-vectors being invariant under Lorentz transformations.

- If we designate the column 4-vector A_μ as a "covariant" vector (where covariant implies that its dot product does not change under Lorentz transformations), then to form a dot product we must multiply by a row vector. Let us write the row 4-vector as A^μ and call it a "contravariant" vector to imply that it is dotted against the covariant vector.
- The label μ on the vector is an index that runs from 0 to 3, specifying the t, x, y, and z components of the 4-vector. Note the convention that when we are indexing a four-vector, we use Greek letters such as μ, ν, etc. , but when we are indexing a three-component vector, we use Latin letters such as i, j, k.
- Using this notation, the dot product of two four-vectors looks like this:

$$\sum_{\mu=0}^{3} A^\mu A_\mu$$

- Note that if we recognize two 4-vectors with the same index as a dot product, the summation symbol is unnecessary. We drop the summation symbol with the understanding that repeated indices always means summation over all values of the index (this is called Einstein notation).

$$A^\mu A_\mu = \sum_{\mu=0}^{3} A^\mu A_\mu$$

- Note that repeated indices only imply summation if they are on the same side of the equals sign. If they are on opposite sides, then repeated indices represent the matching up of components.
- Assume we know the contravariant vector. What does its corresponding covariant vector look like?
- Let us look at the spacetime coordinate 4-vector $x^\mu = (ct, x, y, z)$. We know what its dot product

should look like: $x^{\mu} x_{\mu} = c^2 t^2 - x^2 - y^2 - z^2$

- The only way this is possible is if we define the covariant vector as $x_{\mu} = (ct, -x, -y, -z)$.
- In general then we define the dot product of any two 4-vectors as the product of one in covariant form and the other in contravariant form, where the two forms are related to each other by a sign change of the spatial components:

$$A^{\mu} B_{\mu} = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3 \quad \text{and} \quad A^{\mu} A_{\mu} = A_0^2 - A_1^2 - A_2^2 - A_3^2$$

- Can we mathematically express the relationship between a contravariant vector and its corresponding covariant vector? The metric tensor (similar to a matrix) $g_{\mu\nu}$ accomplishes this.
-Notice that it has two Greek indices, so it is a two-dimensional tensor with 16 components total.

$$x_{\mu} = g_{\mu\nu} x^{\nu} \quad \text{where} \quad g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

- Let us expand this to get a feel for what this notation means:

$$x_{\mu} = \sum_{\nu=0}^{3} g_{\mu\nu} x^{\nu}$$

$$x_{\mu} = g_{\mu 0} x^{(0)} + g_{\mu 1} x^{(1)} + g_{\mu 2} x^{(2)} + g_{\mu 3} x^{(3)}$$

or

$$x_0 = g_{00} x^{(0)} + g_{01} x^{(1)} + g_{02} x^{(2)} + g_{03} x^{(3)} \quad \text{and}$$

$$x_1 = g_{10} x^{(0)} + g_{11} x^{(1)} + g_{12} x^{(2)} + g_{13} x^{(3)} \quad \text{and}$$

$$x_2 = g_{20} x^{(0)} + g_{21} x^{(1)} + g_{22} x^{(2)} + g_{23} x^{(3)} \quad \text{and}$$

$$x_3 = g_{30} x^{(0)} + g_{31} x^{(1)} + g_{32} x^{(2)} + g_{33} x^{(3)}$$

- Plugging in the actual values for the various components of g, these four equations become:

$$x_0 = x^{(0)} \quad \text{and}$$

$$x_1 = -x^{(1)} \quad \text{and}$$

$$x_2 = -x^{(2)} \quad \text{and}$$

$$x_3 = -x^{(2)}$$

- These equations reproduce what we have already said about covariant and contravariant vectors, indicating that we have used the notation correctly.
- We can represent the dot product of 4-vectors A and B in different ways:

$$\begin{aligned} A^{\mu} B_{\mu} = A_{\mu} B^{\mu} \\ &= g_{\mu\nu} A^{\nu} B^{\mu} \\ &= g^{\mu\nu} A_{\nu} B_{\mu} \\ &= A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \\ &= A^0 B^0 - \mathbf{A} \cdot \mathbf{B} \end{aligned}$$

- The covariant tensor $g_{\mu\nu}$ and its corresponding contravariant tensor $g^{\mu\nu}$ *are* related according to:

$g^{\lambda\mu} g_{\mu\nu} = \delta^{\lambda}_{\nu}$ where the identity tensor is defined as $\delta^{\lambda}_{\nu} = 1$ if $\lambda = \upsilon$ and 0 otherwise

- If we think in terms of matrix algebra, all this equation is really saying is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A 4-tensor is the two-dimensional analog of a 4-vector. It is an object that transforms from frame to frame according to the Lorentz transformations and whose complete inner product gives a scalar that is the same in all frames.
- Just like how a 4-vector has one time-like component and three space-like components, a 4-tensor has one time-like row and three space-like rows as well as one time-like column and three space-like columns.
- Consider a 4-tensor F with components:

$$F = \begin{bmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{bmatrix}$$

- The blue component F_{00} is the time-like,time-like component.
- The red components F_{0i} are the time-like,space-like components.
- The green components F_{i0} are the space-like,time-like components.
- The black components F_{ij} are the space-like,space-like components.
- The inner product of two 4-tensors gives us a scalar that is the same in all reference frames.
- The inner product of tensors is equivalent to a 4-vector dot product on both dimensions, so we must flip the sign of the space-like rows and then the space-like columns, leading to:

$$F^{\alpha\beta} F_{\alpha\beta} = F^{00} F^{00} - F^{0i} F^{0i} - F^{i0} F^{i0} + F^{ij} F^{ij}$$

since

$$F^{\alpha\beta}=\begin{bmatrix} F^{00} & F^{01} & F^{02} & F^{03} \\ F^{10} & F_{11} & F_{12} & F^{13} \\ F^{20} & F_{21} & F_{22} & F^{23} \\ F^{30} & F_{31} & F_{32} & F^{33} \end{bmatrix} \quad \text{and} \quad F_{\alpha\beta}=\begin{bmatrix} F^{00} & -F^{01} & -F^{02} & -F^{03} \\ -F^{10} & F_{11} & F_{12} & F^{13} \\ -F^{20} & F_{21} & F_{22} & F^{23} \\ -F^{30} & F_{31} & F_{32} & F^{33} \end{bmatrix}$$

2. Covariant Lorentz Transformation

- With our geometry now defined, we can write the Lorentz transformation in covariant notation as:

$$x'^{\mu}=\Lambda^{\mu}_{\nu}x^{\nu}$$

- The Lorentz transformation tensor Λ transforms the spacetime coordinates x in frame K to the corresponding coordinates x' in frame K'.
- This can be represented in matrix notation as:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}=\begin{bmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \Lambda_{03} \\ \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{20} & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{30} & \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

- For a frame traveling in the x direction, this becomes:

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}=\begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} \quad \text{where} \quad \gamma=\frac{1}{\sqrt{1-v^2/c^2}} \quad \text{and} \quad \beta=v/c$$

- The time-like,time-like component of the Lorentz transformation accounts for pure time dilation.
- The space-like,space-like component accounts for pure length contraction.
- The space-like,time-like and the time-like,space-like components account for temporal and spatial origin shifting, which includes time dilation and length contraction effects.
- Einstein's principle of relativity can now be written:

$$x^{\mu}x_{\mu}=x'^{\mu}x'_{\mu}$$

- Apply the Lorentz transformation to both vectors on the right side:

$$x^{\mu}x_{\mu}=\Lambda^{\mu}_{\nu}x^{\nu}\Lambda^{\lambda}_{\mu}x_{\lambda}$$

- The power of this notation is that the indices preserve the order of operation, which is necessary in matrix algebra, even if we switch the order of the symbols:

$\Lambda^{\mu}_{\nu} x^{\nu} = x^{\nu} \Lambda^{\mu}_{\nu}$ which both mean $\begin{bmatrix} \Lambda_{00} & \Lambda_{01} & \Lambda_{02} & \Lambda_{03} \\ \Lambda_{10} & \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{20} & \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{30} & \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$

- Using this property, we change the dot product of the coordinate 4-vectors to:

$$x^{\mu} x_{\mu} = x^{\nu} \Lambda^{\mu}_{\nu} \Lambda^{\lambda}_{\mu} x_{\lambda}$$

- When first deriving Einstein's relativity, we required all inertial frames to be physically equivalent and that lead to the law of reciprocity. The mathematical statement of reciprocity in 4-vector notation is:

$$\Lambda^{\mu}_{\nu} \Lambda^{\lambda}_{\mu} = \delta^{\lambda}_{\nu}$$

- You can check this for yourself by writing out the Lorentz transformation in matrix form, multiplying it by itself (being careful to use the covariant geometry notation rules) and you end up with the identity matrix.
- Using this relation, we finally have

$$x^{\mu} x_{\mu} = x^{\nu} x_{\nu}$$

- We have arrived at a tautology, indicating that our notation is self-consistent and that the dot product of two 4-vectors is indeed the same in all frames.
- In the language of covariant geometry, we can have covariant/contravariant tensors of different rank, but each transforms from frame to frame according to the Lorentz transformation applied to each dimension.
- The Lorentz operator is applied the number of times equal to the tensor's rank (number of dimensions).
- A rank-zero contravariant tensor is just a scalar and the Lorentz operator is applied zero times, thus a scalar is the same in all frames.
- A rank-one contravariant tensor is a 4-vector with four elements and the Lorentz operator is applied once in the same way it is applied to the coordinate 4-vector:

$$A'^{\mu} = \Lambda^{\mu}_{\nu} A^{\nu}$$

- A rank-two contravariant tensor is a tensor with 16 elements and the Lorentz operator is applied twice to transform to a new frame:

$$F'^{\alpha\beta} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} F^{\mu\nu}$$

3. Covariant Differentiation

- We wish to organize physical properties and mathematical operations into covariant tensors. Once that is accomplished we will know how any other variable transforms simply by constructing it from covariant tensors and applying the rules above.
- Let us start with the partial derivative.

- We already know that the three-component vector of the partial derivative is the gradient:

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$

- The extra component to make a 4-vector must be the time component so that:

$$\frac{\partial}{\partial x^{\mu}} = \left(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \right)$$

$$\boxed{\frac{\partial}{\partial x^{\mu}} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)}$$

- This is commonly written in compact notation as:

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}$$

- This four-dimensional derivative obeys Lorentz transformations and is thus a 4-vector.
- Thus its dot-product does not change under Lorentz transformations:

$$\partial^{\mu}\partial_{\mu} = \frac{\partial}{\partial x_{\mu}}\frac{\partial}{\partial x^{\mu}}$$

$$\partial^{\mu}\partial_{\mu} = \frac{\partial}{\partial x_0}\frac{\partial}{\partial x_0} - \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3}\frac{\partial}{\partial x_3}$$

$$\boxed{\partial^{\mu}\partial_{\mu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2}$$

- Therefore, the wave operator is the same in all inertial reference frames in Special Relativity.
- The derivative 4-vector dotted into a 4-vector A (called the 4-divergence) is:

$$\partial^{\mu} A_{\mu} = \frac{\partial}{\partial x_0} A_0 + \frac{\partial}{\partial x_1} A_1 + \frac{\partial}{\partial x_2} A_2 + \frac{\partial}{\partial x_3} A_3$$

$$\boxed{\partial^{\mu} A_{\mu} = \frac{1}{c}\frac{\partial A_0}{\partial t} + \nabla \cdot \mathbf{A}}$$

- This is the dot product of two 4-vectors and is thus also the same in all frames.

4. Covariant Electrodynamics

- The 4-divergence equation above looks like the charge-current continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

- If we make the identification $A_0 = c\rho$ and $\mathbf{A} = \mathbf{J}$, then we end up with the 4-divergence equation. These components thus form a 4-vector which we call the charge-current 4-vector:

$$\boxed{J^{\mu} = (c\rho, \mathbf{J})}$$

and the continuity equation becomes:

$$\boxed{\partial^{\mu} J_{\mu} = 0}$$

- The wave equation in the Lorenz gauge for the electromagnetic vector potential **A** and scalar potential ϕ are, in Gaussian units:

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\mathbf{A} = \frac{4\pi}{c}\mathbf{J}$$

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\phi = 4\pi\rho \qquad \text{where} \quad \frac{1}{c}\frac{\partial \phi}{\partial t} + \nabla \cdot \mathbf{A} = 0$$

- We already formed the charge and current density into a 4-vector and the wave operator into the dot product of two derivative 4-vectors. The remaining pieces should thus form another four-vector. The Lorenz condition on the right reduces to $\partial^{\mu} A_{\mu} = 0$ if we form the electromagnetic potential 4-vector:

$$\boxed{A^{\mu} = (\Phi, \mathbf{A})}$$

The wave equations then reduce to:

$$\boxed{\partial^{\mu}\partial_{\mu} A^{\alpha} = \frac{4\pi}{c} J^{\alpha}}$$

Wave Equations in Terms of Potentials

- The fields are expressed in terms of the potentials as:

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \Phi$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

- Let us expand these into components and try to use the covariant notation:

$$E_x = -\frac{\partial A_x}{c\partial t} - \frac{\partial \Phi}{\partial x} \quad , \quad E_y = -\frac{\partial A_y}{c\partial t} - \frac{\partial \Phi}{\partial y} \quad , \quad E_z = -\frac{\partial A_z}{c\partial t} - \frac{\partial \Phi}{\partial z}$$

$$E_x = -\frac{\partial A^1}{\partial x_0} + \frac{\partial A^0}{\partial x_1} \quad , \quad E_y = -\frac{\partial A^2}{\partial x_0} + \frac{\partial A^0}{\partial x_2} \quad , \quad E_z = -\frac{\partial A^3}{\partial x_0} + \frac{\partial A^0}{\partial x_3}$$

$$E_x = -(\partial^0 A^1 - \partial^1 A^0) \quad , \quad E_y = -(\partial^0 A^2 - \partial^2 A^0) \quad , \quad E_z = -(\partial^0 A^3 - \partial^3 A^0)$$

and

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \quad , \quad B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \quad , \quad B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

$$B_x = -\frac{\partial A^3}{\partial x_2} + \frac{\partial A^2}{\partial x_3} \quad , \quad B_y = -\frac{\partial A^1}{\partial x_3} + \frac{\partial A^3}{\partial x_1} \quad , \quad B_z = -\frac{\partial A^2}{\partial x_1} + \frac{\partial A^1}{\partial x_2}$$

$$B_x = -(\partial^2 A^3 - \partial^3 A^2) \quad , \quad B_y = -(\partial^3 A^1 - \partial^1 A^3) \quad , \quad B_z = -(\partial^1 A^2 - \partial^2 A^1)$$

- We may be tempted at this point to try to form 4-vectors out of the electric and magnetic field components. But it should be obvious from the forms above that the electric and magnetic field are connected and must be part of the same object. The six components will not fit in a 4-vector, so we must put them in a second-rank covariant tensor. Let us call it the field-strength tensor *F*.
- The six equations above for the six components of the electromagnetic field can each be set to one component of the field-strength tensor.

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 \quad , \quad F^{02} = \partial^0 A^2 - \partial^2 A^0 \quad , \quad F^{03} = \partial^0 A^3 - \partial^2 A^0$$

$$F^{23} = \partial^2 A^3 - \partial^3 A^2 \quad , \quad F^{13} = \partial^1 A^3 - \partial^3 A^1 \quad , \quad F^{12} = \partial^1 A^2 - \partial^2 A^1$$

where $F^{01} = -E_x$, $F^{02} = -E_y$, $F^{03} = -E_z$, $F^{23} = -B_x$, $F^{13} = B_y$, and $F^{12} = -B_z$

- We can now write each component in compact form as:

$$\boxed{F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha}$$

- This approach has told us the six meaningful components of the field strength tensor. But what are the other components?
- First note that switching the order of the indices on *F* just switches the sign of the right hand side, so that $F^{\alpha\beta} = -F^{\beta\alpha}$. This tells us that the tensor is antisymmetric, and we therefore now know six more of its elements. The final four elements are the diagonal elements. Its easy to show:

$$F^{\alpha\alpha} = \partial^\alpha A^\alpha - \partial^\alpha A^\alpha$$

$$F^{\alpha\alpha} = 0$$

- We now know all components of the field strength tensor:

$$\boxed{F^{\alpha\beta}=\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}}$$

- This equation tells us that to find the electromagnetic field in one inertial frame if we know the field in another frame, we apply a Lorentz transformation to both dimensions of the field strength tensor and then reduce it to a set of equations relating electromagnetic field components:

$$F'^{\alpha\beta}=\Lambda^{\alpha}_{\mu}\Lambda^{\beta}_{\nu}F^{\mu\nu}$$

- Writing this equation out in three-vector notation, we have:

$$\mathbf{E}'=\gamma(\mathbf{E}+\boldsymbol{\beta}\times\mathbf{B})-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{E})$$

$$\mathbf{B}'=\gamma(\mathbf{B}-\boldsymbol{\beta}\times\mathbf{E})-\frac{\gamma^2}{\gamma+1}\boldsymbol{\beta}(\boldsymbol{\beta}\cdot\mathbf{B})$$

- Inserting these Lorentz frame transformation rules for the fields into Maxwell's equations, we can reduce the transformed Maxwell's equations down to their original form, proving that Maxwell's equations obey Lorentz transformations.
- In summary, we have formulated electrodynamics in terms of the potentials in covariant form:

$$\boxed{\partial^{\mu}\partial_{\mu}A^{\alpha}=\frac{4\pi}{c}J^{\alpha}}$$ *Wave Equations for Potentials*

$$\boxed{F^{\alpha\beta}=\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha}}$$ *Field-Potential Definitions*

where

$$\boxed{F^{\alpha\beta}=\begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}}\ ,\ \boxed{A^{\mu}=(\Phi,\mathbf{A})}\ ,\ \boxed{J^{\mu}=(c\rho,\mathbf{J})}\ \text{, and }\ \boxed{\partial_{\mu}=\left(\frac{1}{c}\frac{\partial}{\partial t},\nabla\right)}$$

- Alternatively, we can form Maxwell's equations without the use of potentials and find:

$$\boxed{\partial_{\alpha}F^{\alpha\beta}=\frac{4\pi}{c}J^{\beta}}$$

$$\boxed{\partial_{\alpha}F_{\beta\gamma}+\partial_{\beta}F_{\gamma\alpha}+\partial_{\gamma}F_{\alpha\beta}=0}$$