Lecture 12 Notes, Electromagnetic Theory II

C. S. BAIRD
Dr. Christopher S. Baird, faculty.uml.edu/cbaird
University of Massachusetts Lowell
UMASS LOWELL

1. Velocities in Special Relativity

- As was done in Galilean relativity, we can use the coordinate transformation to find out how velocities transform from one frame to the next.
- If frame K' is moving in the x_1 direction at a velocity v relative to frame K, then we found the transformation to be:

$$x_0=\gamma\left(x_0'+\frac{v}{c}x_1'\right) \quad \text{where } x_0 = ct \text{ and } \quad \gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

$$x_1=\gamma\left(x_1'+\frac{v}{c}x_0'\right)$$

- If an object in frame K' moves at some constant velocity u' in the x_1' direction relative to its frame, e.g. a passenger on a moving train walks down the aisle, what we are saying is that its coordinate x_1' changes with respect to time t' at some rate u':

$$u'=\frac{d\,x_1'}{d\,t'} \quad \text{or} \quad u'=c\frac{d\,x_1'}{d\,x_0'} \quad \text{and similarly} \quad u=c\frac{d\,x_1}{d\,x_0}$$

- An incremental displacement of a coordinate dx_1 transforms just like a regular coordinate, so we can transform the incremental distance and time in the u definition according to the Lorentz transformation:

$$u=c\frac{\gamma(d\,x_1'+\frac{v}{c}d\,x_0')}{\gamma\left(d\,x_0'+\frac{v}{c}d\,x_1'\right)}$$

$$u=c\frac{\frac{d\,x_1'}{d\,x_0'}+\frac{v}{c}}{1+\frac{v}{c}\frac{d\,x_1'}{d\,x_0'}}$$

$$u=c\frac{\frac{u'}{c}+\frac{v}{c}}{1+\frac{v\,u'}{c^2}}$$

$$\boxed{u = \frac{u' + v}{1 + \frac{v u'}{c^2}}}$$

- This is the relativistic velocity addition formula for velocities parallel to the direction of frame motion.
- If the speed of the frame v or the speed of the object u' is slow enough, the second term in the denominator approaches zero and the Galilean velocity addition equation is recovered: $u = u' + v$.
- As the speed v of the frame increases, the speed of the object as seen in frame K seems to be slower than what one would expect from Galilean relativity because of length contraction and time dilation.
- Note that this equation tells us that if an object is traveling at the speed of light in one frame, it is traveling at the speed of light in all frames. This behavior is not unique to light, but applies to all objects traveling at the speed of light.
- We must remember that a velocity is defined as a change in position over a change in time. Both position and time are not universal now, so the measured velocity experiences both length contraction and time dilation.
- There is a certain symmetry to this equation that we should expect from the fact that we required that nothing goes faster than the speed of light.
- In the limit that the frame moves at the speed of light, $v \rightarrow c$, the velocity addition formula reduces to $u \rightarrow c$. Therefore, a baseball thrown on a train traveling effectively at c looks from the ground as if it is also traveling effectively at c. In other words, the thrown baseball is effectively stationary, stuck in mid-air, with respect to the train. This makes sense if we remember that when we observe a frame that is traveling effectively at c, its time has effectively stopped. (I say "effectively" because this is only a statement of limiting behavior. No frame can travel at exactly c and time can never perfectly stop.)
- Now consider if the object is moving diagonally in frame K' so that the object has components of its velocity both parallel to and perpendicular to the frame's velocity, $\mathbf{u}' = u_1' \hat{\mathbf{x}}_1' + u_2' \hat{\mathbf{x}}_2'$. For instance, consider a baseball in the moving train that is thrown diagonally up and forward relative to the train (neglect gravity).
- We have already derived the parallel component, and can label it more explicitly:

$$\boxed{u_{\text{par}} = \frac{u_{\text{par}}' + v}{1 + \frac{v u_{\text{par}}'}{c^2}}}$$

Parallel Velocity Addition Formula for Special Relativity

- Now we need to derive how the perpendicular velocity component transforms.
- We may be tempted to say that because $x_2 = x_2'$ we must have $u_2 = u_2'$ as it does in Galilean relativity. But we would be wrong.
- Although it is true that there is no length contraction in directions perpendicular to the frame's velocity, there is still time dilation. All dimensions experience time, so all dimensions experience time dilation. Let us see this mathematically:

$$x_2' = x_2$$

- Take the derivative with respect time in the primed frame:

$$c\frac{dx_2'}{dx_0'}=c\frac{dx_2}{dx_0'}$$

- Expand the derivative on the right to try to get everything on that side in terms of unprimed variables:

$$c\frac{dx_2'}{dx_0'}=c\frac{dx_0}{dx_0'}\frac{dx_2}{dx_0}$$

$$u_2'=\frac{dx_0}{dx_0'}u_2$$

$$u_2'=\gamma\left(1+\frac{vu_1'}{c^2}\right)u_2$$

- Solve for the unprimed variable:

$$u_2=\frac{u_2'}{\gamma\left(1+\frac{vu_1'}{c^2}\right)}$$

- Change to more meaningful labels:

$$\boxed{u_{\text{perp}}=\frac{u_{\text{perp}}'}{\gamma\left(1+\frac{vu_{\text{par}}'}{c^2}\right)}}$$

Perpendicular Velocity Addition Formula for Special Relativity

- We note that the speed of the frame v and the speed of the object in the parallel direction both contribute to time dilation, and therefore both effect the perpendicular speed.
- The object again appears to go slower in the ground frame than in the moving frame.
- If the object is moving completely vertically in the K' frame, so that $u_{\text{par}}' = 0$, this equation reduces to $u_{\text{perp}} = u_{\text{perp}}'/\gamma$ since we have time dilation only coming from the frame's motion.
- If the speed v of the frame approaches the speed of light c, the equation gives $u_{\text{perp}} = 0$ as expected. Again, in this limit, time stops for such a frame and the object seems to hover stationary.

2. Relativistic Momentum and Energy of a Particle

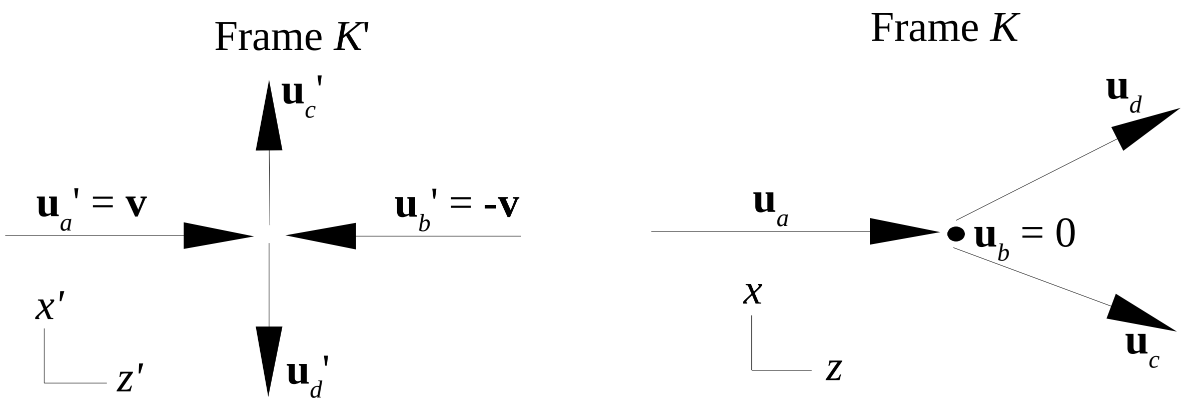
- The classical momentum $\mathbf{p}$ and total energy E of a particle are:

$$\mathbf{p}=m\,\mathbf{u}$$

$$E=E_{u=0}+\frac{1}{2}mu^2$$

where m is the mass, $\mathbf{u}$ is the particle's velocity, and $E_{u=0}$ is some rest energy that is classically

lumped into potential energy.
- We want to find equations of this form that are consistent with the Lorentz transformations.
- We could define the momentum and energy however we want. However, we want them to be meaningful and useful. What made these properties useful in classical mechanics was that they obeyed conservation laws: the conservation of energy and the conservation of momentum.
- To keep them meaningful in Special Relativity, let us require them to still obey conservation laws.
- Consider an elastic collision of two identical particles *a* and *b* that end up as particles *c* and *d*.
- We will define two different inertial frames in which to observe this collision, then we will require the Lorentz transformation to hold between these frames as well as require the conservation laws to hold in both frames.
- The *K*' frame will be the center-of-mass frame and the *K* frame will be the frame where particle *b* is at rest.



- In the center-of-mass frame (K'), symmetry tells us that the initial velocities of *a* and *b* must be equal and opposite, $\mathbf{u}_a' = \mathbf{v}$, $\mathbf{u}_b' = -\mathbf{v}$. The same must hold true for the final velocities, $\mathbf{u}_c' = -\mathbf{u}_d'$. Symmetry also tells us that $u_c' = v$ and $u_d' = v$ as well as $u_c = u_d$.
- We have assumed a 90 degree scattering angle in frame *K*' to make the math easier.
- Note that because of the special way we defined the frames, frame *K* moves with velocity *v* to the left relative to frame *K*' (i.e. frame *K* is effectively hitching a ride on particle *b*). Therefore, the magnitude of the velocity of both particles before and after the collision as measured in *K*' is numerically equal to the velocity *v* of the frame *K*.
- Because of this, if we have expressions that depend only on velocities as measured in frame *K*, we should be able to get the expressions as functions of only *v* by Lorentz transforming to frame *K*'.
- The conservation laws in both frames state:

Frame *K*': $\mathbf{p}_a'+\mathbf{p}_b'=\mathbf{p}_c'+\mathbf{p}_d'$, $E_a'+E_b'=E_c'+E_d'$

Frame *K*: $\mathbf{p}_a+\mathbf{p}_b=\mathbf{p}_c+\mathbf{p}_d$, $E_a+E_b=E_c+E_d$

- Because the masses are identical before and after the collision, the masses will cancel out everywhere in the equations. We can therefore consider momentum and energy to be undetermined functions of velocity alone.
- Applying what we know about the velocities based on symmetry alone, the conservation laws

in both frames become:

Frame K': $\mathbf{p}(\mathbf{v})+\mathbf{p}(-\mathbf{v})=\mathbf{p}(\mathbf{u}_c')+\mathbf{p}(-\mathbf{u}_c')$, $E(v)+E(v)=E(v)+E(v)$

Frame K: $\mathbf{p}(\mathbf{u}_a)=\mathbf{p}(\mathbf{u}_c)+\mathbf{p}(\mathbf{u}_d)$, $E(u_a)+E(u_b=0)=E(u_c)+E(u_c)$

- The energy conservation equation in frame K' is trivially satisfied because of the way we set up the frame and because we have identical particles. This equation will not give us any useful information.
- The self-consistent nature of spacetime leads us to assume **p**(-v) = -**p**(v). Applying this relation to the momentum expression in frame K', we find that this expression is also trivially satisfied and will not give us any information.
- We are only left with the expressions in frame K, which simplify to:

$$\mathbf{p}(\mathbf{u}_a)=\mathbf{p}(\mathbf{u}_c)+\mathbf{p}(\mathbf{u}_d) \quad , \quad E(u_a)+E_0=2E(u_c)$$

- Let us only look at the z component of the momentum. Note that in frame K, the momentum of particle a only has a z component so that the z component equals the total momentum. Also note that both particles have the same final z component of the momentum in this frame due to symmetry:

$$p(\mathbf{u}_a)=2\,p_z(\mathbf{u}_c) \quad , \quad E(u_a)+E_0=2E(u_c)$$

- We want to get rid of p_z in favor of p. We can do so by noting that the momentum vector and the corresponding velocity vector point in the same direction, so that the z component of the momentum relates to the total momentum in the same way as the velocity:

$$\frac{p_z}{p}=\frac{u_{c,z}}{u_c}$$

$$p_z=p\frac{u_{c,z}}{u_c}$$

- Inserting this into the conservation of momentum equation in frame K, we now have:

$$p(\mathbf{u}_a)=2\,p(\mathbf{u}_c)\frac{u_{c,z}}{u_c} \quad , \quad E(u_a)+E(u_b=0)=2E(u_c)$$

- Since all the velocities in frame K' have the value v which is also the frame velocity, if we Lorentz transform to this frame, our conservation laws should only be a function of v.
- We generally Lorentz transform a velocity by using the velocity addition formula:

$$u_{\text{par}}=\frac{u_{\text{par}}'+v}{1+\dfrac{v\,u_{\text{par}}'}{c^2}}$$

- Apply this equation to particle a, remembering that $u_a' = v$ to find:

$$\boxed{u_a = \frac{2v}{1+\frac{v^2}{c^2}}}$$

- Similarly, apply the parallel velocity addition formula to the particle after the collision (particle *c*), noting that the final particle has no parallel component to its velocity $u_{c,\text{par}}' = 0$:

$$\boxed{u_{c,z} = v}$$

- To find how the *x* component of the velocity of the final particle relates in both frames, we must use the perpendicular velocity addition formula:

$$u_{\text{perp}} = \frac{u_{\text{perp}}'}{\gamma\left(1+\frac{v u_{\text{par}}'}{c^2}\right)}$$

$$u_{c,x} = \frac{u_{c,x}'}{\gamma\left(1+\frac{v u_{c,z}'}{c^2}\right)}$$

- Noting that $u_{c,z}' = 0$ and $u_{c,x}' = v$, this becomes:

$$\boxed{u_{c,x} = v\sqrt{1-v^2/c^2}}$$

- The magnitude of u_c is the square root of the sum of the square of its components:

$$u_c = \sqrt{u_{c,x}^2 + u_{c,z}^2}$$

$$\boxed{u_c = v\sqrt{2-v^2/c^2}}$$

- Applying the velocity transformation rules (the equations above in boxes) to the conservation laws, they become:

$$p\left(\frac{2v}{1+\frac{v^2}{c^2}}\right) = 2\,p\left(v\sqrt{2-v^2/c^2}\right)\frac{1}{\sqrt{2-v^2/c^2}} \quad , \quad E\left(\frac{2v}{1+\frac{v^2}{c^2}}\right) + E(0) = 2\,E\left(v\sqrt{2-v^2/c^2}\right)$$

- We now have everything in terms of *v* and can solve for *p* and *E*.
- Solving these types of problems is not straightforward. Like a differential equation, the best we can do is guess a general form and then see if it works.
- Because the momentum must reduce to the classical form $p = mu$ for $u \ll c$, and motivated by previous results, let us try a solution for the momentum of the form:

$$p(u) = \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} g(u) \quad \text{where } g(u \to 0) = 1$$

- Applying this to the conservation law, we find:

$$\frac{m\left(\frac{2v}{1+v^2/c^2}\right)}{\sqrt{1-\frac{1}{c^2}\left(\frac{2v}{1+v^2/c^2}\right)^2}} g(u_a) = 2\frac{mv\sqrt{2-v^2/c^2}}{\sqrt{1-\frac{1}{c^2}\left(v\sqrt{2-v^2/c^2}\right)^2}} g(u_c) \frac{1}{\sqrt{2-v^2/c^2}}$$

- After simplifying this equation, we end up with: $g(u_a) = g(u_c)$
- The velocities u_a and u_c are potentially different and arbitrary, so we must have:

$$g(u) = 1 \quad \text{for all } u$$

- This is true for any velocity u so that the final solution is:

$$\boxed{\mathbf{p} = \frac{m\mathbf{u}}{\sqrt{1-\frac{u^2}{c^2}}}} \quad \text{or} \quad \boxed{\mathbf{p} = \gamma_u m\mathbf{u}}$$

- As a particle is observed to go faster, it is also observed to gain more momentum than expected by Galilean relativity.
- For the energy equation try the form

$$E(u) = \frac{E_0}{\sqrt{1-u^2/c^2}} h(u) \quad \text{as a trial solution to} \quad E\left(\frac{2v}{1+\frac{v^2}{c^2}}\right) + E(0) = 2E\left(v\sqrt{2-v^2/c^2}\right)$$

where $h(0) = 1$ to ensure that we end up with the rest energy E_0 if there is no kinetic energy.
- Applying this trial solution, we find:

$$\left(\frac{E_0}{\sqrt{1-1/c^2\left(\frac{2v}{1+v^2/c^2}\right)^2}}\right) h(u_a) + E_0 = 2\left(\frac{E_0}{\sqrt{1-v^2(2-v^2/c^2)/c^2}}\right) h(u_c)$$

$$\left(\frac{1+v^2/c^2}{1-v^2/c^2}\right) h(u_a) + 1 = 2\left(\frac{1}{1-v^2/c^2}\right) h(u_c)$$

$$\left(1+v^2/c^2\right)\left(h(u_a)-1\right) = 2\left(h(u_c)-1\right)$$

- Again, the velocities u_a and u_c are potentially different and arbitrary, so that both sides must disappear independently. This is only possible if:

$$h(u)=1 \quad \text{for all } u$$

- This equality must be true for all velocities u so that we have our solution:

$$E(u)=\frac{E_0}{\sqrt{1-u^2/c^2}}$$

- But what is the rest energy E_0? Note that we cannot derive it using an elastic collision like the one we have formulated here because there is no change in rest energy so that rest energy is conserved trivially.
- Instead, we find E_0 by requiring the Special Relativity expression for energy to reduce to the Galilean expression at low speeds.
- Use the Taylor series expansion $(1-x)^{-1/2}=1+\frac{1}{2}x+\frac{3}{8}x^2+...$ and apply it to the relativistic energy expression to find:

$$E=E_0\left(1+\frac{1}{2}\frac{u^2}{c^2}+\frac{3}{8}\frac{u^4}{c^4}+...\right)$$

- For low speeds ($u << c$) we have $u/c << 1$ and we can drop all terms except the first two:

$$E=E_0+E_0\frac{1}{2}\frac{u^2}{c^2}$$

- We force this to equal the classical result $E=E_0+\frac{1}{2}mu^2$ and solve for E_0:

$$E_0+\frac{1}{2}mu^2=E_0+E_0\frac{1}{2}\frac{u^2}{c^2}$$

$$\boxed{E_0=mc^2}$$

- This is Einstein's famous mass-energy equivalence which allows for the annihilation of mass by converting it to energy.
- With E_0 known, the final energy expression becomes:

$$\boxed{E=\frac{mc^2}{\sqrt{1-u^2/c^2}}} \quad \text{or} \quad \boxed{E=\gamma_u mc^2}$$

- A particle that is observed to go faster is also observed to get much more kinetic energy than the classical expression would predict.
- This is essentially what prohibits us from accelerating a particle with mass to the speed of light. The closer we push it to the speed of light, the more the energy we give the particle just becomes additional kinetic energy instead of additional speed.
- In the limit that the particle is accelerated to the speed of light ($u \rightarrow c$), its energy becomes

infinite. In other words, it takes an infinite amount of energy to accelerate a particle with mass to exactly the speed of light.

3. Energy-Momentum 4-vector

- If we can take physical properties like momentum and energy and form them into 4-vectors, then we can present the mathematics more cleanly.
- Because the dot-product of two 4-vectors is the same in all frames (a Lorentz invariant), and because the laws of physics are the same in all frames, we should be able to present the laws of physics as the dot-product of two 4-vectors (or higher-order, Lorentz-covariant mathematical entities).
- The velocity of an object depends on its change in spatial position divided by change in time. Both spatial positions and time are affected by Lorentz transformations, so we cannot expect the velocity to change from one frame to the next according to the exact form of the Lorentz transformations. Indeed, we have already shown above that the velocity addition formulas are derived from the Lorentz transformations but have a different form.
- However, if we multiply the traditional velocity of the object by the factor γ_u, then we are mathematically compensating for time dilation and would expect the resulting object to transform according to Lorentz transformations. This is indeed the case.
- We define the velocity 4-vector U as

$$\boxed{U=(U_{0,}\mathbf{U})=(\gamma_u c\,,\,\gamma_u \mathbf{u})}$$

which indeed transforms from frame to frame according to Lorentz transformations. In other words, directly applying a Lorentz transformation to the velocity 4-vector reproduces the relativistic velocity addition formula (it is left to the interested reader to do this).
- With the 4-velocity defined in this way, the momentum and energy can be defined in terms of the components of the 4-velocity.

$$\mathbf{p}=m\,\mathbf{U} \quad \text{and} \quad \frac{E}{c}=m\,U_0$$

- This shows that we can now form a energy-momentum 4-vector P which is the mass m times the velocity 4-vector:

$$\boxed{P=\left(\frac{E}{c}\,,\mathbf{p}\right)} \quad \text{so that} \quad \boxed{P=m\,U=(m\,U_{0,}m\,\mathbf{U})=(m\,c\,\gamma_u\,,\,m\,\mathbf{u}\,\gamma_u)}$$

- The dot product of two 4-vectors is Lorentz invariant (independent of frame):

$$P\cdot P=P'\cdot P'$$

$$P_0^2-|\mathbf{p}|^2=P'^2_0-|\mathbf{p}'|^2$$

$$\frac{E^2}{c^2}-p^2=m^2\gamma_u^2c^2-m^2\gamma_u^2u^2$$

$$\frac{E^2}{c^2} - p^2 = m^2 c^2$$

$$\boxed{E^2 = p^2 c^2 + m^2 c^4}$$

- This is true in all frames.