Lecture 11 Notes, Electromagnetic Theory II

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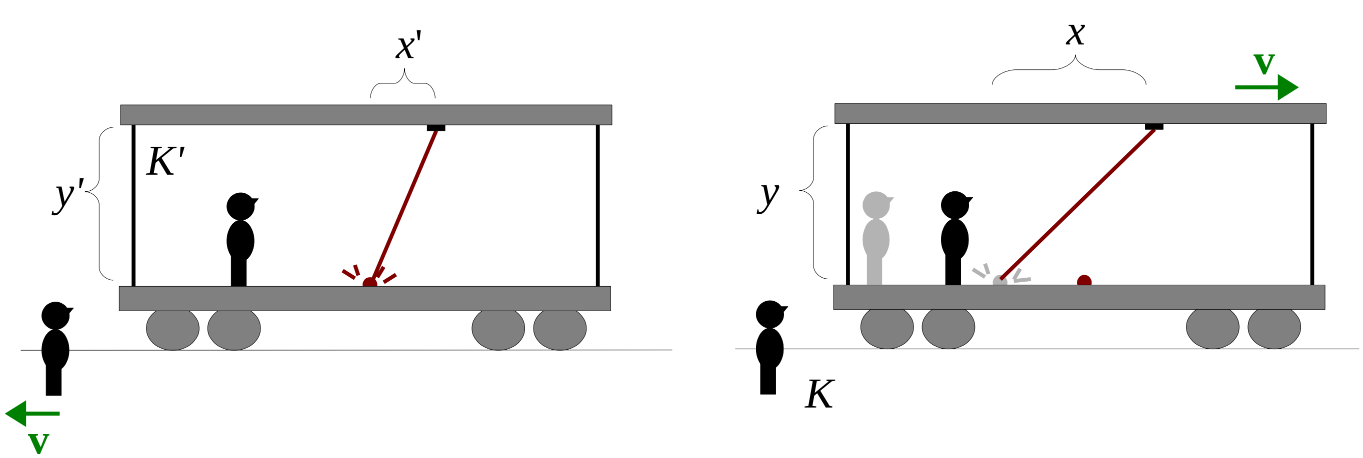
1. Einstein's Principle of Relativity

- The goal is to find a principle of relativity between different frames of reference that will make Maxwell's equations have the same form in all frames.
- Einstein began with the two postulates:
    1. The laws of physics, including Maxwell's equations, are the same in all inertial frames.
    2. The speed of light is the same in all inertial frames (this postulate is suggested by Maxwell's equations, which can be turned into a wave equation with a free-space wave speed *c* that is the same in all frames).
- Absent from these postulates is any assumption of universal time or universal lengths.
- We will derive Einstein's Special Relativity using a case-by-case conceptual approach. A more fundamental derivation using the structure of spacetime is presented in the Appendix at the end of this document.
- Consider train car (frame K') moving at some constant speed v with respect to the ground (frame K).
- In frame K', at time zero, a light bulb on the car's floor is turned on and at time t' a detector detects its light. In the frame K', the detector has a fixed displacement of (x', y', z') relative to the bulb.
- In frame K, the detector moved along with the car and the light hitting it must have traveled a different path and taken a time t to do so.



- First let us state that the light bulb and the train are used simply to help us picture the physics. We are really talking about any two events in spacetime and their relationship to each other, so our results will be general.
- Align the spatial and temporal origin of both frames to each other and to the *event A* (the bulb flashing). Then *event B* (the detector detecting the light) is observed to happen in frame K' at (x', y', z', t') and in frame K at (x, y, z, t).
- Let us calculate the speed of light in each frame and use the second postulate to force these speeds to be equal to the universal constant c.

- In calculating the speed, we make use of the first postulate and say that speed is defined as distance over time in both frames, as measured in its own frame.
- In frame *K'* we have:

$$c=\frac{\sqrt{x'^2+y'^2+z'^2}}{t'}$$

- In the lab frame *K* we have:

$$c=\frac{\sqrt{x^2+y^2+z^2}}{t}$$

- Simplify both equations to find:

$$c^2 t'^2-x'^2-y'^2-z'^2=0 \quad \text{and} \quad c^2 t^2-x^2-y^2-z^2=0$$

- Since they are equal to zero, they are certainly equal to each other, leading to:

$$\boxed{c^2 t'^2-(x'^2+y'^2+z'^2)=c^2 t^2-(x^2+y^2+z^2)}$$ *Principle of Special Relativity*

- This is the fundamental principle of Einstein's Special Relativity relating the coordinates in two different inertial frames. It is essentially a mathematical version of the two postulates.
- Historically, Einstein called his theory of relativity for inertial frames the "Special Theory of Relativity" to differentiate it from his theory for non-inertial frames.
- For simplicity, assume the frame motion is in the *x* dimension. Thus, the frames are static in the *y* and *z* dimensions relative to each other, so these coordinates are automatically equal.

$y'=y$ and $z'=z$ leading to:

$$c^2 t'^2-x'^2=c^2 t^2-x^2$$

- The equation above is not complete. There are two unknowns (x' and t') but only one equation.
- There must be another equation to get a unique solution.
- The other equation is the velocity of frame *K'* relative to *K* in terms of the spacetime dimensions:

$$v=v(x,t,x',t')$$

- Once we know this velocity equation, we can combine it with the relativity principle and uniquely solve for the primed variables in terms of the unprimed ones (or vice versa).

2. Lorentz Transformations

- The general form of the frame velocity equation is complicated and not very enlightening.
- Let us instead focus on deriving the velocity equation for a few special cases, and then we will build up the results into the general solution.

- For the first special case, consider that the events *A* and *B* occur at the same point in x' location in frame *K'* but at different times (the detector is fixed right above the bulb in frame *K'*).

- Therefore $x' = 0$. In this case, the lateral displacement of event B from the origin is completely due to the motion of the frame traveling at speed v, so that:

$$v=\frac{x}{t}$$

- The relativity principle with $x' = 0$ for this special case reduces down to:

$$c^2 t'^2 = c^2 t^2 - x^2$$

- Substitute the velocity into the above equation and solve for the time:

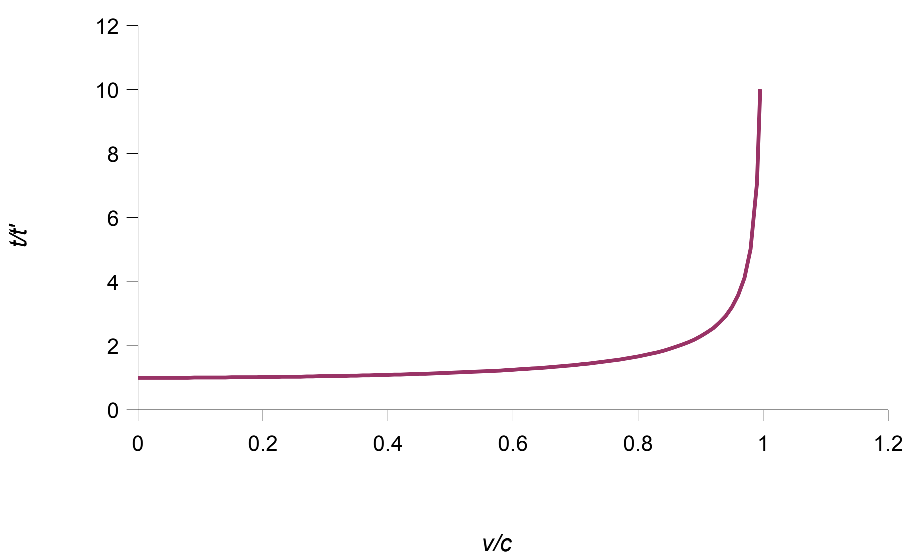
$$c^2 t'^2 = c^2 t^2 - v^2 t^2$$

$$t' = \sqrt{1-\frac{v^2}{c^2}}\, t$$

- Invert this equation to find:

$$\boxed{t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\, t'}$$

Time Dilation



- First note that if the velocity of the frame v is greater than the speed of light c, the equation above becomes the square root of a negative number, which is imaginary. Since time is not imaginary-valued, this tells us that a frame can never go faster than the speed of light. Since an object can always be taken to be at rest in some frame, this tells us that an object can never go faster than the speed of light.

- Since the velocity v is always less than c, the denominator is always between 0 and 1, and t is always greater than t'.

- If I am on the ground and I look in the train, I see the clocks running slow and the time between events larger than if the car were stationary. The faster the train travels, the more I see its time dilated. (Time itself runs slower; there is nothing wrong with the clocks.)
- In the limit that the train approaches the speed of light, I see its time infinitely dilated. In other words, time slows to a stop.
- Note that for speeds less than 0.5*c*, the plot above shows us that $t/t' = 1$ to an excellent approximation, meaning that $t = t'$. This was the behavior of time according to Galilean relativity. In other words, for object speeds less than about 300 million miles per hour, time dilation is so small that it can typically be ignored. Most of the speeds we experience in everyday life are much smaller than this speed.
- In general, the equations of Special Relativity should always reduce down to the Galilean-Newton equations in the low-speed limit. This is a good way to error check our equations.
- The first postulate tells us that the laws of physics are the same in all inertial frames, and therefore there is no privileged frame. Thus, it is just as valid to say that the ground is the moving frame and the train is the rest frame.
- Therefore, if I am on the train, I see my own clocks running normally and I see the clocks on the ground running slow.
- Time dilation is best summed up as:

A moving clock runs slow as measured by the the stationary observer

- The person on the train sees the ground frame as having slow clocks, while the people on the ground see the train as having the slow clocks. This seems to be a paradox; which one is *really* going slow? However, it is not a paradox because there is no absolute time and no absolute reference frame in Special Relativity. There is no “really”.

- For the next special case, let us consider measuring the length of the train. The observer in frame K' measures it to be x' and the observer in frame K measures it to be x.
- If the observer in K' defines t' as the amount of time it takes for some object fixed in frame K (e.g. a lamppost) to move from the front of the car to the back of the car, he can make the relation:

$$x' = vt'$$

- In the exact same way, the observer in K on the ground can define the time t as the time it takes for the train to pass him from front to back so that:

$$x = vt$$

- From these equations it becomes apparent that:

$$\frac{x'}{t'} = \frac{x}{t}$$

$$x = \frac{t}{t'} x'$$

- We already know the time dilation relationship t/t', but we have to be careful because the train (whose length we are measuring) is at rest in the *K*' frame, so that in this case *K* is the moving frame, so that we must swap prime and unprimed variables in the time dilation equation:

$$\frac{t}{t'}=\sqrt{1-\frac{v^2}{c^2}}$$

Plugging the above equation into the previous one we find:

$$\boxed{x=\sqrt{1-\frac{v^2}{c^2}}\,x'}$$

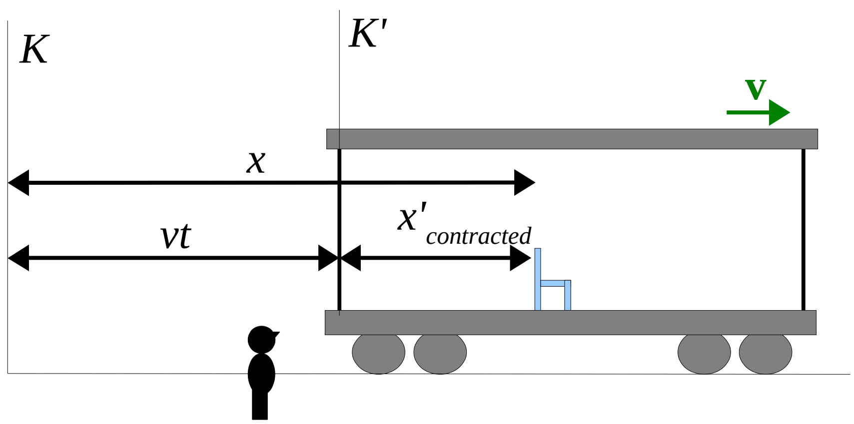
Length Contraction

- The number under the square root is always less than one, so that the length of the object *x* is always less than *x*'.
- If I am on the train, the train has its normal length. However, when the observer on the ground measures the train, he finds it shorter than when it was parked. (There's nothing wrong with the train; space itself is contracted.)
- The faster the train goes, the shorter it gets in the direction of travel as seen by the ground observer.
- Length contraction is best summed up as:

A moving ruler is shortened as measured by the stationary observer

- Note that people on the train see the ground as moving and therefore see ground objects as shortened, while people on the ground see the objects on the train as shortened. Again, this is not a paradox because there is no universal frame.

- With the two special cases having shown us how lengths and times transform, we can now combine them and add origin shifting to get the general transformation equations, known as the Lorentz transformation.
- A chair on the train is measured to reside at a distance *x*' in frame *K*' from the origin of this frame, which we could say is at the back of the train.
- The location of the chair in frame *K* is *x,* which is just the sum of the location of the train plus the location of the chair relative to the train, which must be contracted in this frame.



$$x = vt + x'_{contracted}$$

$$x = vt + \sqrt{1 - \frac{v^2}{c^2}}\, x'$$

$$x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}(x - vt)$$

- Plug the above equation directly into $c^2 t'^2 - x'^2 = c^2 t^2 - x^2$ and solve for t' to find:

$$t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\left(t - \frac{vx}{c^2}\right)$$

- In summary, the general solution was found to be:

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned} \qquad \text{where} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}}$$

Lorentz Transformation

- If we desired to go the other way, from the primed variables to the unprimed ones, we could invert these equations. But we don't need to. In Special Relativity, there are no special frames of reference. The equations above must be the exact same ones to use if we are going from the primed variables to the primed ones, with the exception of the velocity. We defined the velocity as positive in one frame, so it would be negative in the other frame. In practice, this means we take the equations above, make every primed variable unprimed, make every unprimed variable primed, and replace v with $-v$.
- With time and space no longer universal, it makes more sense to speak of events rather than just locations or times.
- Time can be thought of as a fourth coordinate describing events.
- If we are going to treat time as a fourth coordinate, the mathematics will be cleaner if it has the same dimensions as the spatial coordinates.
- We do this (if we are determined not to abandon SI units) by multiplying time by c so that:

$$x_0 = ct, \;\; x_1 = x, \;\; x_2 = y, \;\; x_3 = z$$

- With these definitions, the principle of relativity becomes:

$$x_0'^2 - (x_1'^2 + x_2'^2 + x_3'^2) = x_0^2 - (x_1^2 + x_2^2 + x_3^2)$$

$$x_0'^2 - |\mathbf{x}'|^2 = x_0^2 - |\mathbf{x}|^2$$

- Also, with these definitions, the Lorentz transformation becomes:

$$x_0' = \gamma\left(x_0 - \frac{v}{c}x_1\right)$$
$$x_1' = \gamma\left(x_1 - \frac{v}{c}x_0\right) \qquad \text{where } \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$
$$x_2' = x_2$$
$$x_3' = x_3$$

- The Lorentz transformation can be thought of as a combination of <u>length contraction</u>, <u>time dilation</u>, and a <u>shifting of the origins</u> relative to each other. But note that the shifting of origins effects both the space and time coordinates.
- For instance, suppose two events happen at the same time, $t = 0$ as seen in frame K, but one event is at the spatial origin and the other event is at location x. Since the events are simultaneous in frame K, any time discrepancy between the events as seen in frame K' will be solely due to temporal origin shifting. Plugging these values into the Lorentz transformation, we find $t' = -vx\gamma/c^2$. Therefore, the event at location x happens before the event at the origin as seen frame K' because of temporal origin shifting.
- In general, two events that are simultaneous in one frame will not be simultaneous in any other frame. This is called the "relativity of simultaneity".

3. Four-Vectors

- With the time coordinate considered as a fourth spatial coordinate, it becomes very useful to formally define a four-dimensional vector which specifies an event in space-time <u>*and*</u> transforms from frame to frame according to the Lorentz transformation.
- "Four-Vector" is the name that we give to this definition.

Four-Vector: A four-dimensional vector obeying Lorentz transformations

- Any four-dimensional vector that obeys the Lorentz transformation is a four-vector.
- A four-vector has three space-like components and one time-like component that transform under Lorentz transformations exactly like the space and time dimensions.
- The coordinate four-vector (x_0, x_1, x_2, x_3) is the most basic four-vector.
- If a four-vector (A_0, A_1, A_2, A_3) obeys the Lorentz transformation, it must also obey the relativity principle from which the transformation was derived:

$$A_0'^2 - |\mathbf{A}'|^2 = A_0^2 - |\mathbf{A}|^2$$

- From this we see that the dot product of any two four-vectors is invariant under Lorentz transformations (i.e. is the same in all frames).

$$A_0' B_0' - \mathbf{A}' \cdot \mathbf{B}' = A_0 B_0 - \mathbf{A} \cdot \mathbf{B}$$

- The dot product of two four-vectors is not what you would expect if you simply took the dot product of a three-dimensional vector and added a dimension.
- Because of the negative sign between the spatial components and the time component of the relativity principle, we must have a relative negative sign in the definition of the four-vector dot product if it is to behave like dot products should, i.e. be invariant under transformations.

APPENDIX

A more fundamental and elegant way to derive the Lorentz transformation is to apply basic assumptions about the properties of spacetime and then require a universal speed limit. Assume that frame *K'* moves at speed *v* in the *x* direction relative to frame *K* so that we can ignore the *y* and *z* dimensions. The homogeneity of spacetime requires that points in space-time have everywhere the same density as observed in one particular frame. Mathematically, spacetime homogeneity therefore dictates linear relationships between all space and time coordinates. If, for instance, $x' = x^3$, than at locations marked by higher *x*, successive points in *x'* would be farther and farther apart. A linear relationship means that the coordinates relations must have the form:

$$x'=f\,x+f_2\,t \qquad t'=g\,t+h\,x$$

At this point, f, f_2, g, and h are arbitrary functions that cannot depend on *x* or *t*. The only thing left that they could depend on is the frame's velocity *v* and universal constants.
We have inertial frames in relative motion, so that the spatial shifting of the origin must be taken into account. In Galilean relativity, origin shifting was taken into account by using a term -*vt*. We can anticipate an origin shifting effect by taking a factor -*v* out of the arbitrary functions f_2 and *h*:

$$x'=f\,x-v\,f_2\,t \qquad t'=g\,t-v\,h\,x$$

The homogenous nature of spacetime also means that as time marches on, points in space cannot spread out. This forces upon us $f_2=f$, leading to:

$$x'=f\,x-v\,f\,t \qquad t'=g\,t-v\,h\,x$$

Since the functions *f*, *g*, *h* may be functions of *v*, they must be functions of the magnitude of *v* and not depend on the sign of *v* in order to preserve origin shifting. The most natural way ensure that a function depends only on the magnitude of a variable is to square the variable. We therefore assume that if these functions depend at all on the velocity, they will depend on v^2:

$$x'=f(v^2)\,x-v\,f(v^2)\,t \qquad t'=g(v^2)\,t-v\,h(v^2)\,x$$

The isotropic nature of spacetime means that there is no special direction. The behavior of spacetime is the same no matter which direction the frame is moving. Therefore, we can switch the primed variables with the unprimed variables and the equations above should still hold (as long as we are careful to realize that the frame is now moving in the opposite direction, so we must switch -*v* to +*v*). This leads to:

$$x=f\,x'+v\,f\,t' \qquad t=g\,t'+v\,h\,x'$$

Now take the expressions for *x*' and *t*' and insert them into the two equations above. A basic requirement of any good theory is internal self-consistency. By switching the labels and combining the equations, we are enforcing self-consistency. This leads to:

$$x=f(fx-vft)+vf(gt-vhx) \qquad t=g(gt-vhx)+vh(fx-vft)$$

Collecting terms in *x* and *t*, we find:

$$x=(f^2-v^2fh)x+(g-f)fvt; \qquad t=(g^2-v^2hf)t+(f-g)hvx$$

In order for the first equation to reduce down to $x = x$ and the second equation to reduce down to $t = t$, we must have

$$f^2-v^2fh=1 \quad \text{and} \quad f=g$$

which leads to:

$$h=\frac{f^2-1}{fv^2}$$

so that now the coordinate relations have been reduced down to:

$$x'=f(x-vt) \qquad t'=f\left(t-vx\frac{f-1/f}{fv^2}\right)$$

or equivalently:

$$x=f(x'+vt') \qquad t=f\left(t'+vx'\frac{f-1/f}{fv^2}\right)$$

Now all we need to do is find is *f*. Notice that these expressions are still quite general. In fact, these expressions become Galilean relativity if we set $f = 1$. All we have done so far is require spacetime to be homogenous, isotropic, self-consistent, and preserve origin shifting. In order to find *f*, we apply the postulate of a universal speed limit. The velocity of an object is just the change of its position with respect to time:

$$u=\frac{dx}{dt}$$

The quantity *dx* is an incremental displacement in *x* and transforms in the exact same way as *x*. The same is true of *dt*. Applying the above transformation rules to both increments, we find:

$$u=\frac{f(dx'+vdt')}{f\left(dt'+vdx'\frac{(f-1/f)}{fv^2}\right)}$$

Divide top and bottom by *dt*' and recognize that $\frac{dx'}{dt'}=u'$ to find:

$$u = \frac{u' + v}{1 + v u' \frac{(f - 1/f)}{f v^2}}$$

We now require there to be a universal limiting speed *c* that no physical object in general can exceed. What that means is that if an object is going *c* in one frame, it must be going *c* in the other frame, as it can go no faster, so that $u = c$ and $u' = c$:

$$c = \frac{c + v}{1 + v c \frac{(f - 1/f)}{f v^2}}$$

Solve for *f* to find:

$$f = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Now that we have found the final unknown, we have the full solution, which is the Lorentz transformation:

$$x' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} (x - v t) \qquad t' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \left(t - \frac{v x}{c^2}\right)$$

Experimental measurements reveal that the universal limiting speed limit *c* is indeed the speed of light in vacuum. We thus see that it is not necessary to talk about light at all to derive the Lorentz transformation, as long as we require a universal speed limit. If someday light was found to travel slightly slower than c, this finding would have no effect on the theory of Special Relativity, since fundamentally Special Relativity depends on there being a universal speed limit, and not on light traveling at that speed limit.