# Lecture 1 Notes, Electromagnetic Theory II

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## 1. Overview of the Course

- Last semester we covered electrostatics, magnetostatics, pseudo-magnetostatics, and introductory electrodynamics.
- This semester we will study electrodynamics in depth as well as special relativity.
- In electrodynamics, changing magnetic fields can give rise to changing electric fields which in turn can give rise to new magnetic fields. This feedback process continues indefinitely and a self-sustaining electromagnetic wave propagates and becomes independent of any electric charges or currents.
- Seen from a physical perspective, every *electrodynamic* system necessarily involves electromagnetic waves in one form or another, whether being created, destroyed or transmitted.
- We will therefore focus this semester on:
    - The interaction of waves with materials (reflection, refraction, dispersion, absorption)
    - Bounded electromagnetic waves (waveguides, cavities)
    - The creation of electromagnetic waves (radiation, antennas, etc.)
    - The interaction of waves with objects (scattering)
    - Special relativity
- In this course, any electrodynamic field will be referred to as a "wave" or as "light". Although in many contexts the word "light" narrowly refers to visible light, it is used in this course to mean any electrodynamic field of any frequency (e.g. radio waves, microwaves, X-rays, etc.) and even those without well-defined frequencies. Similarly, any electrodynamic field is referred to as a "wave", even if the wave is not traveling (e.g. standing waves, evanescent waves).
- It should be noted that waves that are on the high-frequency end of the spectrum (gamma rays, X-rays, ultraviolet, and often even visible) have such small wavelengths that they are often better described using quantum electrodynamics when interacting with materials.

## 2. Review of Maxwell's Equations

- The behavior of classical electrodynamic fields are completely described by Maxwell's equations:

$$\nabla\cdot\mathbf{D}=\rho \quad , \qquad \nabla\cdot\mathbf{B}=0$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \quad , \qquad \nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}$$

- The free electric charge density $\rho$ gives rise to a diverging electric field **D**.
- The change of the total magnetic field **B** in time gives rise to a curling total electric field **E**.
- The total magnetic field **B** is always non-diverging (there are no magnetic charges).
- The free electric current density **J** as well as the change of the electric field **D** in time give rise to a curling magnetic field **H**.
- These equations cannot be used until the material's response to the electromagnetic fields is known, summarized by finding a relationship between **D** and **E** as well as between **B** and **H**.
- In free space (vacuum), $\mathbf{D} = \varepsilon_0\mathbf{E}$ and $\mathbf{H} = (1/\mu_0)\mathbf{B}$.

- In linear materials, $\mathbf{D} = \varepsilon\mathbf{E}$ and $\mathbf{H} = (1/\mu)\mathbf{B}$.
- Most materials behave linearly to a good approximation.
- The electric permittivity $\varepsilon$ and magnetic permeability $\mu$ in general depend on spatial coordinates and on time.
- The non-uniform temporal response of a material to electromagnetic waves is described by the permittivity's and permeability's dependence on angular wave frequency $\omega$. This measures the frequency at which a wave oscillates in time as it travels through the material.
- The flow of energy is described by the energy density flow vector, known as the Poynting vector **S**:

$$\mathbf{S}=\mathbf{E}\times\mathbf{H}$$

- Due to the way it was derived, the Poynting vector only has strict meaning in approximately linear, lossless materials.
- Note that the expression for the Poynting vector shown above has the implicit meaning that the real parts of the fields are used. A more explicit expression of the Poynting vector is:

$$\mathbf{S}=[\Re(\mathbf{E})]\times[\Re(\mathbf{H})]$$

- Whenever you see $\mathbf{S} = \mathbf{E}\times\mathbf{H}$, you should immediately tell yourself that this expression is telling you to take the reals parts of the fields first
- Note that there many other expressions that we can derive for the Poynting vector if we apply various averaging techniques or assumptions. However, these expressions are less general. It is better to use the more general expression shown above.
- The total energy density $u$ is given by:

$$u=\frac{1}{2}(\mathbf{E}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{H})$$

- These obey the law of conservation of energy:

$$\frac{\partial u}{\partial t}=-\nabla\cdot\mathbf{S}-\mathbf{J}\cdot\mathbf{E}$$

## 3. Plane Waves in Materials that are Linear, Uniform, Isotropic, but Frequency-Dependent

- To study the interaction of waves with materials, it makes sense to start with the simplest waves (single-frequency transverse plane waves) and the simplest materials (linear, uniform, isotropic, frequency-dependent materials with neither charges nor currents present).
- Single-frequency transverse planes waves are very useful because we can always build up more complex waves as a Fourier superposition of the plane waves.
- We will see that single-frequency transverse plane waves are the particular solutions to Maxwell's equations in linear, uniform, isotropic, source-free materials, and the general solution is just the superposition of all possible frequencies, polarizations, and directions, weighted by coefficients.
- We start with Maxwell's equations in the absence of sources ($\rho = 0$, $\mathbf{J} = 0$):

$$\nabla\cdot\mathbf{D}=0 \quad , \qquad \nabla\cdot\mathbf{B}=0$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \quad , \qquad \nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}$$

- Linear, uniform, isotropic, frequency-dependent materials obey: $\mathbf{D} = \varepsilon(\omega)\mathbf{E}$ and $\mathbf{H} = (1/\mu(\omega))\,\mathbf{B}$.
- Plugging this into the source-free Maxwell equations gives us:

$$\nabla\cdot\mathbf{E}=0 \quad , \qquad \nabla\cdot\mathbf{B}=0$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \quad , \qquad \nabla\times\mathbf{B}=\mu(\omega)\epsilon(\omega)\frac{\partial\mathbf{E}}{\partial t} \tag{1}$$

- Take the curl of the bottom equations and use the identity $\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$ :

$$\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}+\frac{\partial}{\partial t}\nabla\times\mathbf{B}=0 \quad \text{and} \quad \nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}-\mu(\omega)\epsilon(\omega)\frac{\partial}{\partial t}\nabla\times\mathbf{E}=0$$

- Now use the non-diverging nature of **E** and **B** as stated in the top equations to drop out the first terms:

$$\nabla^2\mathbf{E}-\frac{\partial}{\partial t}\nabla\times\mathbf{B}=0 \quad \text{and} \quad \nabla^2\mathbf{B}+\mu(\omega)\epsilon(\omega)\frac{\partial}{\partial t}\nabla\times\mathbf{E}=0$$

- Now substitute in the curl of the fields as given in the original lower equations (1), which leads to:

$$\boxed{\nabla^2\mathbf{E}-\mu(\omega)\epsilon(\omega)\frac{\partial^2\mathbf{E}}{\partial t^2}=0} \quad \text{and} \quad \boxed{\nabla^2\mathbf{B}-\mu(\omega)\epsilon(\omega)\frac{\partial^2\mathbf{B}}{\partial t^2}=0}$$

- We have decoupled the fields and ended up with wave equations.
- Note that we have only mathematically decoupled the electric and magnetic fields. Physically, they are still coupled together according to the original Maxwell equations.
- Wave equations typically have sines and cosines as solutions.
- Let us try solutions that are of the form:

$$\mathbf{E}=\mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\omega t} \quad \text{and} \quad \mathbf{B}=\mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{x}}e^{-i\omega t}$$

- It should be noted that we use the complex representation of sines and cosines because this form is more manageable mathematically. (Remembering that Euler's equation relates the two forms according to $e^{ix}=\cos x+i\sin x$ .)
- However, imaginary numbers do not exist in the real world, so to get a final expression for the fields, we must always take the real part of the complex-valued fields.
- The property **k**, known as the wave vector, describes the spatial frequency (number of radians of a cycle of oscillation per unit length), as well as the direction in which the waves travel. The magnitude of the wavevector **k** is referred to as the wavenumber $k$.
- An alternate way of describing the scale of the spatial variation is the wavelength $\lambda$.

- Both descriptions are equivalent and are related through $k = 2\pi/\lambda$.
- Plugging in our trial solutions gives:

$$-k^2 \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}} + \mu(\omega)\epsilon(\omega)\omega^2 \mathbf{E}_0 e^{i\mathbf{k}\cdot\mathbf{x}} = 0 \quad \text{and} \quad -k^2 \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{x}} + \mu(\omega)\epsilon(\omega)\omega^2 \mathbf{B}_0 e^{i\mathbf{k}\cdot\mathbf{x}} = 0$$

$$-k^2 + \mu(\omega)\epsilon(\omega)\omega^2 = 0$$

$$\boxed{k = \pm\sqrt{\mu(\omega)\epsilon(\omega)}\,\omega}$$

- We have found that the harmonic wave is indeed the solution to the wave equations, but only if the spatial nature of the wave (the wavenumber $k$) is related to the temporal nature of the wave (the frequency $\omega$) as shown above.
- Equations that connect the spatial and temporal aspects of waves inside materials are known as "dispersion relations" because they describe how different frequencies get dispersed.
- The positive $k$ value is the one that typically gives results consistent with experiments. (The negative value corresponds to exotic waves that propagate in meta-materials.)
- From this point on, we assume we are dealing with standard materials and use the positive $k$.
- If the material's permittivity and permeability can be approximated to be frequency-independent, then the dispersion relation becomes:

$$\boxed{k = \sqrt{\mu\epsilon}\,\omega}$$

- The wavenumber $k$ and frequency $\omega$ are now related by a constant and no dispersion occurs.
- Furthermore, in a vacuum, the relation becomes:

$$\boxed{k = \sqrt{\mu_0\epsilon_0}\,\omega}$$

- From regular wave mechanics (e.g. waves on a string) we know that the wavenumber and angular frequency are related through the wave's phase velocity $v_p$ according to:

$$v_p = \frac{\omega}{k}$$

- We can thus solve for the phase velocity of electromagnetic waves in a uniform material to find:

$$v_p = \frac{1}{\sqrt{\mu(\omega)\epsilon(\omega)}}$$

- In vacuum this becomes:

$$v_p = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c$$

- A numerical calculation of the velocity yields $c = 3\times10^8$ m/s.
- When Maxwell first did this derivation, he was astonished to see that the speed of his newly discovered electromagnetic waves matched the speed of light.

- This was a historical turning point in physics. This discovery united electrodynamics and optics, showing that radio waves and visible light are the same thing and obey the same laws.
- Thus the speed of light in vacuum $c$ and the electric permittivity $\varepsilon_0$ and magnetic permeability $\mu_0$ of free space are all universal constants, and can be used interchangeably.
- The velocity of light in any material is always slower than the velocity of light in free space.
- Physically, this means that the electric charges in the material interact with the wave and slow it down.
- Because the speed of light $c$ in free space is the ultimate speed limit, it makes sense to express the speed of light $v_p$ in a material in terms of $c$.

$$v_p = \frac{c}{c} \frac{1}{\sqrt{\mu(\omega)\epsilon(\omega)}}$$

$$v_p = \frac{c}{\sqrt{\frac{\mu(\omega)\epsilon(\omega)}{\mu_0 \epsilon_0}}} = \frac{c}{\sqrt{\mu_r \epsilon_r}}$$

$$\boxed{v = \frac{c}{n}} \quad \text{where} \quad \boxed{n = \sqrt{\frac{\mu(\omega)\epsilon(\omega)}{\mu_0 \epsilon_0}} \quad \text{or} \quad n = \sqrt{\mu_r \epsilon_r}}$$

- The variable $n$ is know as the index of refraction. If we solve for $n$ to find $n = c/v$ we see that it is a dimensionless variable greater than one that measures the extent to which a material slows down waves.
- The variable $\mu_r$ is the relative magnetic permeability, $\mu_r = \mu/\mu_0$ , and the variable $\epsilon_r$ is the relative electric permittivity, $\epsilon_r = \epsilon/\epsilon_0$ , which is also called the "dielectric constant".
- Quite often we deal with materials that are dielectric but are not significantly magnetic, so that $\mu_r = 1$. In this case, the index of refraction becomes:

$$\boxed{n = \sqrt{\epsilon_r}} \quad \text{in non-magnetic materials}$$

- Often the dielectric constant $\epsilon_r$ is a complex-valued number and this relation is more involved than it may seem at first.
- When a wave enters a new medium and changes speed, its path bends or "refracts".
- The "index of refraction" gets its name because it can be measured by observing how much the wave's path refracts when entering the material.
- In summary, the principle solution of the wave equations in simple materials is a single-frequency plane wave:

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \quad \text{and} \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)} \quad \text{where} \quad k = \pm\sqrt{\mu(\omega)\epsilon(\omega)}\,\omega$$

- These waves are called plane waves because at a certain time $t$, the fields have a constant value everywhere on each plane perpendicular to $\mathbf{k}$.
- The general solution to the wave equation is an infinite sum over all plane wave solutions, weighted by coefficients. In this way it becomes obvious that an arbitrary wave shape can be constructed using a sum of plane waves if the coefficients are chosen properly.
- When we transformed Maxwell's equations into the wave equations, we lost information about the directional part of the wave vector $\mathbf{k}$ as well as the relationship between the electric and

magnetic field. We can now recover this information by plugging the plane wave solutions back into Maxwell's equations. This gives us:

$$\mathbf{k}\cdot\mathbf{E}_0=0 \quad , \quad \mathbf{k}\cdot\mathbf{B}_0=0$$

$$\mathbf{k}\times\mathbf{E}_0=\omega\,\mathbf{B}_0 \quad , \quad \mathbf{k}\times\mathbf{B}_0=-\omega\,\mu(\omega)\epsilon(\omega)\mathbf{E}_0$$

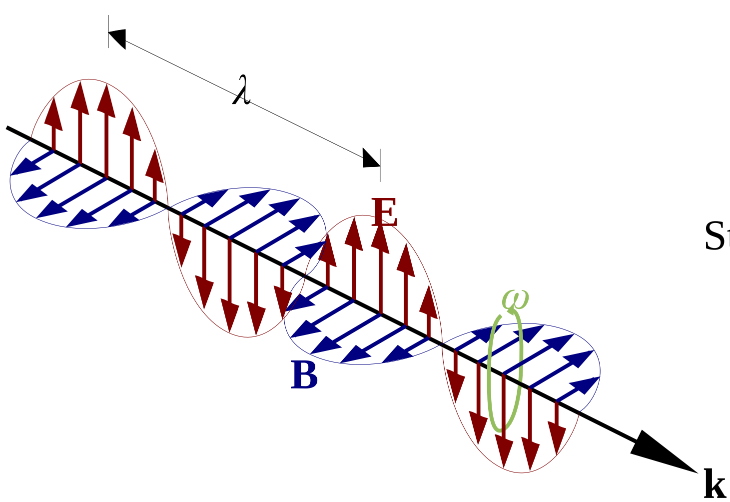
- From these equations it becomes obvious that the electric field vector **E**, magnetic field vector **B**, and the propagation vector **k** are all perpendicular to each other according to a right-handed triple.
- Note that if the wave is traveling in a meta-material, the dispersion relation has an extra overall negative factor as mentioned previously. This negative sign shows up in the above equations, causing the fields to obey a left-handed triple. For this reason, meta-materials are also called “left-handed” materials.
- It must be remembered that this derivation is only valid inside uniform materials away from sources. In a region of non-uniformity, such as at the boundary between materials, or near sources, the vectors behave very differently.
- Using the dispersion relation that we found previously, and equating vector magnitudes, the bottom equations above both reduce to the same result:

$$\boxed{E_0=\frac{1}{\sqrt{\mu(\omega)\epsilon(\omega)}}B_0}$$ for linear, uniform materials

- This can also be written as:

$$\boxed{E_0=\frac{c}{n}B_0}$$

- If $n$ is real-valued, then the equation above tells us that the electric field and the magnetic field have the same phase. In other words, the electric field and magnetic field reach a peak at the same moment, reach zero at the same moment, etc.
- If $n$ is complex-valued, then the equation above tells us that the electric field and magnetic field are permanently out of phase by some factor determined by the phase of $n$.




Light Propagation in Standard Materials (Right-Handed Materials)

## 4. Energy of Plane Waves

- As discussed earlier, the Poynting vector describes the nature of how electromagnetic energy flows and is defined as:

$$\mathbf{S}=\mathbf{E}\times\mathbf{H} \quad \text{which really means } \mathbf{S}=[\Re(\mathbf{E})]\times[\Re(\mathbf{H})]$$

- Let us apply this to plane waves to see how energy flows in plane waves in linear uniform materials.
- We must remember to take the *real parts of the fields* before taking the cross product.
- We substitute in the solutions for the fields:

$$\mathbf{S}=(\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x}-\omega t))\times(\frac{1}{\mu}\sqrt{\mu\epsilon}\,\hat{\mathbf{k}}\times\mathbf{E}_0\cos(\mathbf{k}\cdot\mathbf{x}-\omega t))$$

$$\mathbf{S}=\frac{1}{\mu}\sqrt{\mu\epsilon}\,E_0^2\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega t)\,\hat{\mathbf{E}}_0\times(\hat{\mathbf{k}}\times\hat{\mathbf{E}}_0)$$

$$\boxed{\mathbf{S}=\sqrt{\frac{\epsilon}{\mu}}\,E_0^2\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega t)\,\hat{\mathbf{k}}} \quad \text{for transverse, monochromatic, plane waves in simple materials.}$$

- This is the instantaneous energy density flow (the "instantaneous Poynting vector"). We can see that energy always flows in the direction of propagation for right-handed materials, in the positive **k** direction. We also see that the rate at which it flows oscillates.
- Often of more use is not the instantaneous energy flux but the time-averaged energy flux, which is known as the "irradiance" or the "intensity" $I$ and is equivalent to power per area. This parameter arose because early optical detectors which measured electromagnetic energy flux could not respond fast enough to give the instantaneous $S$, but instead outputted the time-averaged value.
- We take a time average by integrating over some time interval and dividing by time:

$$<\mathbf{S}>=\frac{\int_{t-T/2}^{t+T/2}\mathbf{S}(t')\,dt'}{\int_{t-T/2}^{t+T/2}dt'}$$

$$<\mathbf{S}>=\sqrt{\frac{\epsilon}{\mu}}\,E_0^2\frac{1}{T}\hat{\mathbf{k}}\int_{t-T/2}^{t+T/2}\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega t)\,dt'$$

$$<\mathbf{S}>=\sqrt{\frac{\epsilon}{\mu}}\,E_0^2\frac{1}{T}\hat{\mathbf{k}}\frac{1}{\omega}\int_{\mathbf{k}\cdot\mathbf{x}-\omega t-\omega T/2}^{\mathbf{k}\cdot\mathbf{x}-\omega t+\omega T/2}\cos^2(u)\,du$$

$$<\mathbf{S}>=\frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}\,E_0^2\hat{\mathbf{k}}\left[1+\frac{1}{2}\cos(2\mathbf{k}\cdot\mathbf{x}-2\omega t)\left(\frac{\sin(\omega T)}{\omega T}\right)\right]$$

- The time interval $T$ must be sufficiently large for the averaging to be useful. As $T$ gets large, the function $(\sin(\omega T)/\omega T)$ approaches zero. This leaves:

$$<\mathbf{S}>=\frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}E_0^2\hat{\mathbf{k}}$$ *Intensity of a Simple Plane Wave in Simple Materials*

- Thus on average, energy flows constantly in the direction of propagation and is proportional to the square of the electric field maximum $E_0$. Note that the expression above is very limited and is only valid for a single, monochromatic, transverse, traveling plane wave in uniform, linear materials. Usually, it is better to use the general expression, $\mathbf{S} = \mathbf{E}\times\mathbf{H}$.

## 5. Wave Polarization

- As we have shown, the magnetic field in plane waves follows the electric field in a very predictable way. For this reason, we typically talk about the electric field and ignore the magnetic field. It is always there and we can find the magnetic field whenever we want to. Don't let the fact that we ignore the magnetic field so much make you think it is not there.
- We now know that plane waves have an electric field of the form:

$$\mathbf{E}=\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

- We have said nothing of the polarization vector $\mathbf{E}_0$ other than that we know it is independent of time and space, and lies in a plane perpendicular to the propagation direction $\mathbf{k}$.
- Here the polarization (polar wave orientation) $\mathbf{E}_0$ of a wave should not be confused with the polarization $\mathbf{P}$ of the dielectric material that the wave induces.
- Let us expand the wave's polarization vector $\mathbf{E}_0$ into the most general form we can.
- Because we know this vector must lie in the plane perpendicular to $\mathbf{k}$, it can only have two components:

$$\mathbf{E}=(\boldsymbol{\epsilon}_1 E_1+\boldsymbol{\epsilon}_2 E_2)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

- Here $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$ are linear unit coordinate vectors that define the plane perpendicular to $\mathbf{k}$, and $E_1$ and $E_2$ are the vector's components in this coordinate system. For instance, if we align our rectangular coordinate axes with the wave, it could have the form $\mathbf{E}=(\hat{\mathbf{x}}E_x+\hat{\mathbf{y}}E_y)e^{i(kz-\omega t)}$.
- In general, the components may also be complex numbers. Let us expand them in terms of magnitude and phase.

$$\mathbf{E}=(\boldsymbol{\epsilon}_1|E_1|e^{i\theta_1}+\boldsymbol{\epsilon}_2|E_2|e^{i\theta_2})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

- Here, $\theta_1$ and $\theta_2$ are the phases of the complex numbers $E_1$ and $E_2$ respectively. We can distribute to get a feel for what these mean.

$$\mathbf{E}=\boldsymbol{\epsilon}_1|E_1|e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_1)}+\boldsymbol{\epsilon}_2|E_2|e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_2)}$$ *General Form for the Polarization of a Plane Wave*

- The simplest polarization state is when there is only one component present. This happens when the wave is linearly polarized in the direction of one of the unit vectors. For example if $E_2 = 0$ then:

$$\mathbf{E}=\boldsymbol{\epsilon}_1|E_1|e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_1)}$$ *Horizontal Polarization*

- This is often referred to as *horizontally* polarized light.

- If only the other component is present, then we have *vertically* polarized light:

$$\mathbf{E}=\boldsymbol{\epsilon}_2|E_2|e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_2)} \quad \textit{Vertical Polarization}$$

- The general form for the polarization can be thought of as the superposition of two linearly polarized waves that are perpendicular.
- If both components are not zero, but have the same phase, $\theta_1 = \theta_2$, then the wave reduces to:

$$\mathbf{E}=(\boldsymbol{\epsilon}_1|E_1|+\boldsymbol{\epsilon}_2|E_2|)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_1)} \quad \textit{Linear Polarization}$$

- This is a linearly polarized wave where the direction of polarization lies at an angle $\tan^{-1}(|E_2|/|E_1|)$ from the $\boldsymbol{\epsilon}_1$ axis.
- If the components have phases that are not equal, $\theta_1 \neq \theta_2$, then the wave is elliptically polarized. The point where the electric field is always at its peak traces out an elliptical spiral in space.
- A special case of elliptical polarization is if $\theta_2 = \theta_1 \pm \pi/2$ and $|E_1| = |E_2|$. This is known as circular polarization.

$$\mathbf{E}=|E_1|(\boldsymbol{\epsilon}_1 \pm i\boldsymbol{\epsilon}_2)e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_1)} \quad \textit{Circular Polarization}$$

- With circularly polarized light, the peak traces out a circle in space. The upper sign gives a counterclockwise rotating wave ("left circular") when viewed facing into the oncoming wave, and the lower sign gives a clockwise rotating wave ("right circular").
- This can best be understood by aligning the propagation direction with the direction *z*, the basis vector $\boldsymbol{\epsilon}_1$ with the direction *x* and $\boldsymbol{\epsilon}_2$ with the direction *y*, and taking the real part of the components to get the actual fields:

$$E_x=|E_1|\cos(kz-\omega t+\theta_1) \quad , \quad E_y=\mp|E_1|\sin(kz-\omega t+\theta_1)$$

- The magnitude of the total electric field vector for circularly polarized light is:

$$|\mathbf{E}|=\sqrt{E_x^2+E_y^2}$$

$$|\mathbf{E}|=|E_1|\sqrt{\cos^2(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_1)+\sin^2(\mathbf{k}\cdot\mathbf{x}-\omega t+\theta_1)}$$

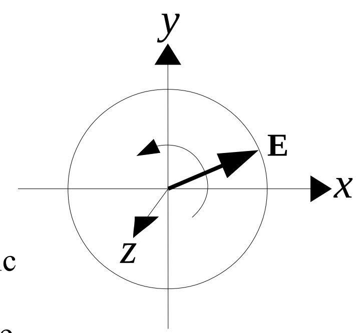
$$|\mathbf{E}|=|E_1|$$

- Thus the magnitude of the electric field **E** vector stays constant at its peak value $|E_1|$ over all space and time. This is the definition of circle.
-We must be careful with the language that we use here and what we mean by it. The magnitude of a vector and the magnitude of a complex number are totally different things even though they may use the same absolute value symbol. The electric field is both a *vector* and a *complex number*, so we must understand which magnitude we really mean. The above equation should be read as, "the vector-magnitude of the real part of the electric field at all points in space equals the complex-number-magnitude of the field strength."

- The angle $\theta$ that the total electric field vector makes with the $x$-axis is:

$$\theta = \tan^{-1}\left(\frac{E_y}{E_x}\right)$$

$$\theta = \mp(kz - \omega t + \theta_1)$$




- Thus at the origin at $t = 0$, the electric field vector makes an angle $\pm\theta_1$ to the $x$ axis. As time passes, the angle that the electric field makes with the $x$ axis gets steadily more positive for the upper sign (left-circular polarization), and more negative for the lower sign (right-circular polarization).
- The two circularly polarized waves are also a valid a set of orthogonal basis vectors. We define:

$$\boldsymbol{\epsilon}_+ = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 + i\boldsymbol{\epsilon}_2) \quad \text{and} \quad \boldsymbol{\epsilon}_- = \frac{1}{\sqrt{2}}(\boldsymbol{\epsilon}_1 - i\boldsymbol{\epsilon}_2)$$

- The general form for a plane wave in terms of these basis vectors becomes:

$$\mathbf{E} = (\boldsymbol{\epsilon}_+ E_+ + \boldsymbol{\epsilon}_- E_-)e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

$$\boxed{\mathbf{E} = \boldsymbol{\epsilon}_+ |E_+| e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t + \theta_+)} + \boldsymbol{\epsilon}_- |E_-| e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t + \theta_-)}}$$

## 6. Measuring Polarization

- In practice, it is much more difficult to directly measure the phases of a plane wave than the magnitudes, especially at high frequencies.
- We can measure the phases indirectly by measuring the magnitude multiple times (or the field intensities which are just the squares of the magnitudes) while using polarizing filters.
- The Stokes parameters do just that. They are intensity parameters that are easily measurable and can be used to find the polarization state. For fully polarized waves, they are defined as follows:

$$s_0 = |E_1|^2 + |E_2|^2 = |\mathbf{E}|^2$$

$$s_1 = |E_1|^2 - |E_2|^2$$

$$s_2 = 2\,\Re(E_1^* E_2)$$

$$s_3 = 2\,\Im(E_1^* E_2)$$

- The Stokes parameters $s_0$, $s_1$, $s_2$, $s_3$, are also often labeled $I$, $Q$, $U$, $V$. Note that Stokes parameters can describe completely-polarized as well as mixed-polarization light.
- The Stokes parameters are formed into a column vector called the Stokes vector:

$$\mathbf{S} = \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ s_3 \end{bmatrix}$$

- The Stokes parameters can be easily inverted to give the complex field values.
- An object's scattering properties can be summarized in an intensity scattering matrix known as a Mueller matrix *M*.
- The scattered wave (as represented by the scattered Stokes vector) then equals the scattering matrix times the incident wave's Stokes vector: $\mathbf{S}_{\text{scat}} = M\,\mathbf{S}_{\text{inc}}$.
- The zeroeth Stokes parameter, $s_0$, is just the overall intensity.
- The first Stokes parameter, $s_1$, is the difference of the principal components in the linear basis.
- The second Stokes parameter, $s_2$, is the difference of the principal components in the linear basis rotated 45°.
- The last Stokes parameter, $s_3$, is the difference of the principal components in the circular basis.
- Note that the four Stokes parameters are not independent. You can easily prove $s_0^2 = s_1^2 + s_2^2 + s_3^2$. The one missing piece of information, the overall phase of the wave, is not as important in practice because it is dependent on the origin. One can always make it zero by appropriately selecting the origin.
- The Stokes parameter approach is used when the phase is hard to measure directly, typically at high frequencies such as with visible light.
- At microwave and radio frequencies, the phase can be measured directly more easily.
- In applications that involve these wavelengths, such as in radar imaging, the Jones vector **J** is used instead of the Stokes vector to represent the polarization state:

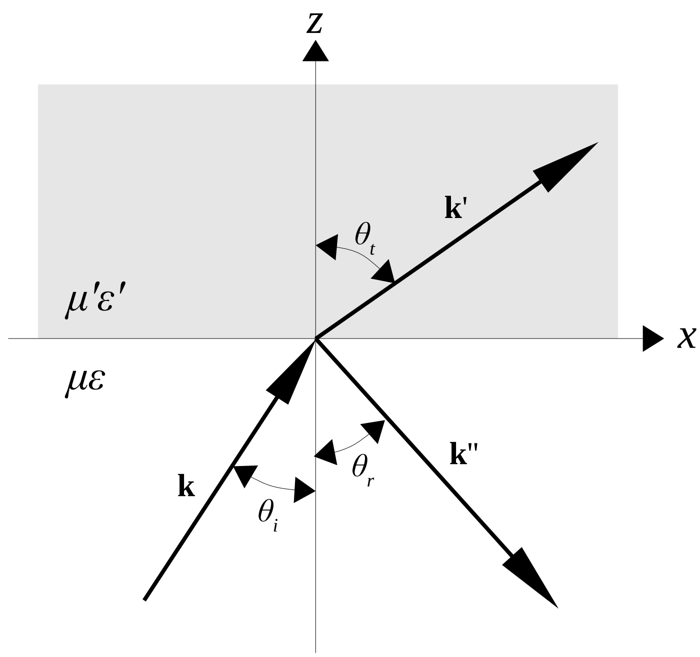
$$\mathbf{J} = \begin{bmatrix} |E_1| e^{i\theta_1} \\ |E_2| e^{i\theta_2} \end{bmatrix} \quad \text{or} \quad \mathbf{J} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}$$

- Similar to the 4×4 intensity scattering matrix (the Mueller matrix), the 2×2 complex-valued scattering matrix (the Jones matrix) sums up the scattering properties of an object. The scattered Jones vector equals the object's Jones matrix times the incident Jones vector: $\mathbf{J}_{\text{scat}} = S\,\mathbf{J}_{\text{inc}}$.

## 7. Reflection and Refraction

- Now that we know what electromagnetic waves look like in an infinite uniform linear material, let us add one more level of complexity.
- What if we have two semi-infinite uniform linear materials that meet at a perfectly flat boundary surface. There are no sources and no boundary conditions at infinity. What do the electromagnetic fields look like?
- The material is not spatially uniform across the whole problem, so the wave equation does not apply to this problem in a general sense.
- But within each semi-infinite region, the material is uniform. The wave equation with its plane wave solution as derived above applies within in each region. We are left needing to just apply the boundary conditions where the two semi-infinite materials meet.
- Let us set the planar boundary that is between the materials so that it coincides with the *x*-*y* plane at $z = 0$.
- The material filling the space $z < 0$ has electric permittivity $\varepsilon$ and magnetic permeability $\mu$, and

thus an index of refraction of $n=\sqrt{\mu\epsilon/\mu_0\epsilon_0}$ .
- The material filling the space $z > 0$ has electric permittivity $\varepsilon'$ and magnetic permeability $\mu'$, and thus an index of refraction $n'=\sqrt{\mu'\epsilon'/\mu_0\epsilon_0}$ .
- To be as general as possible, both indices of refraction (and therefore all corresponding wavevectors) are complex-valued. This allows the possibility of lossy materials.
- The incident wave **E** comes from negative $z$ at some angle $\theta_i$ relative to the $z$ axis.
- The transmitted/refracted wave **E**' goes towards positive $z$ at some angle $\theta_t$ relative to the $z$ axis.
- The reflected wave **E**" goes back towards negative $z$ at some angle $\theta_r$ relative to the $z$ axis.
- These are real geometric angles, not to be confused with the complex phase angles of the waves.
- The "plane of incidence" is the plane that contains all of the wavevectors (it is the plane of the page in the drawing below). It is not to be confused with the plane separating the two materials.




- The mathematical expressions for the plane waves are therefore:

$$\text{Incident:} \quad \mathbf{E}=\mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$$

$$\text{Transmitted:} \quad \mathbf{E}'=\mathbf{E}_0' e^{i(\mathbf{k}'\cdot\mathbf{x}-\omega' t)}$$

$$\text{Reflected:} \quad \mathbf{E}''=\mathbf{E}_0'' e^{i(\mathbf{k}''^{*}\cdot\mathbf{x}-\omega'' t)}$$

- Note that there is a complex conjugate on the reflected wavevector **k**". Textbook derivations neglect this operation and end up with equations which are only valid if the initial material is non-lossy (in other words, the textbook equations are only valid when **k**" is real such that **k**"* = **k**"). However, Maezawa and Miyauci showed (J. Opt. Soc. Am. A/Vol. 26, No. 2/February 2009) that this conjugation must be included in order for the final equations to obey conservation of energy when the initial material is lossy. Conceptually, the conjugation means that the wave is *growing* in the +$z$ direction due to material losses, but this makes sense when one remembers that the reflected wave is traveling in the -$z$ direction. The complex conjugation on **k**" is simply carried through the rest of the derivation to get the more general solutions. The general solutions derived here reduce down to the equations shown in Jackson when you

assume non-lossy initial material so that **k**"* becomes **k**" and therefore $n^*$ becomes $n$.
- The boundary conditions must hold for all time and all points on the boundary. This means that the exponentials must match at $z = 0$, leading to:

$$\omega = \omega' = \omega''$$

$$(\mathbf{k}\cdot\mathbf{x})_{z=0} = (\mathbf{k}'\cdot\mathbf{x})_{z=0} = (\mathbf{k}''^{*}\cdot\mathbf{x})_{z=0}$$

- This immediately tells us that all of the wave vectors lie in the same plane, called the plane of incidence. Let us look at the second equation more closely:

$$(k_x x + k_z z)_{z=0} = (k_x' x + k_z' z)_{z=0} = (k_x''^{*} x + k_z''^{*} z)_{z=0}$$

$$k_x = k_x' = k_x''^{*}$$

- The system is homogenous in the $x$ direction, so the fields cannot depend on the $x$ coordinate. As a result, $k_x''$ must be real:

$$k_x = k_x' = k_x''$$

$$k\sin\theta_i = k'\sin\theta_t = k''\sin\theta_r$$

- Now using the dispersion relations, $k = \sqrt{\mu\epsilon}\,\omega$ , $k' = \sqrt{\mu'\epsilon'}\,\omega$ , and $k'' = \sqrt{\mu\epsilon}\,\omega$ :

$$\sqrt{\mu\epsilon}\,\omega\sin\theta_i = \sqrt{\mu\epsilon}\,\omega\sin\theta_r$$

$$\boxed{\theta_i = \theta_r}$$ *The Law of Reflection*

$$\sqrt{\mu\epsilon}\,\omega\sin\theta_i = \sqrt{\mu'\epsilon'}\,\omega\sin\theta_t$$

$$\boxed{n\sin\theta_i = n'\sin\theta_t}$$ *The Law of Refraction* or *Snell's Law*

- Note that this means that waves always bend towards the normal when entering a higher-index material, and away from the normal when leaving a higher-index material.
- The four boundary conditions when no charges or currents are present are:

$$[\mathbf{D}_2\cdot\mathbf{n} = \mathbf{D}_1\cdot\mathbf{n}]_{\text{on } S} \quad\rightarrow\quad [\mathbf{D}'\cdot\hat{\mathbf{z}} = (\mathbf{D}+\mathbf{D}'')\cdot\hat{\mathbf{z}}]_{z=0} \qquad \text{(B.C. 1)}$$

$$[\mathbf{B}_2\cdot\mathbf{n} = \mathbf{B}_1\cdot\mathbf{n}]_{\text{on } S} \quad\rightarrow\quad [\mathbf{B}'\cdot\hat{\mathbf{z}} = (\mathbf{B}+\mathbf{B}'')\cdot\hat{\mathbf{z}}]_{z=0} \qquad \text{(B.C. 2)}$$

$$[\mathbf{E}_2\times\mathbf{n} = \mathbf{E}_1\times\mathbf{n}]_{\text{on } S} \quad\rightarrow\quad [\mathbf{E}'\times\hat{\mathbf{z}} = (\mathbf{E}+\mathbf{E}'')\times\hat{\mathbf{z}}]_{z=0} \qquad \text{(B.C. 3)}$$

$$[\mathbf{H}_2\times\mathbf{n} = \mathbf{H}_1\times\mathbf{n}]_{\text{on } S} \quad\rightarrow\quad [\mathbf{H}'\times\hat{\mathbf{z}} = (\mathbf{H}+\mathbf{H}'')\times\hat{\mathbf{z}}]_{z=0} \qquad \text{(B.C. 4)}$$

- We put each boundary condition in terms of the total electric field using $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{H} = \mathbf{B}/\mu$ and $\mathbf{k}\times\mathbf{E} = \omega\mathbf{B}$ and realize that the time- and position-dependent parts all cancel out leaving only the polarization vectors:

$$\epsilon'\mathbf{E}_0'\cdot\hat{\mathbf{z}}=(\epsilon\,\mathbf{E}_0+\epsilon\,\mathbf{E}_0'')\cdot\hat{\mathbf{z}} \qquad \text{(B.C. 1)}$$

$$(\mathbf{k}'\times\mathbf{E}_0')\cdot\hat{\mathbf{z}}=((\mathbf{k}\times\mathbf{E}_0)+(\mathbf{k}''^*\times\mathbf{E}_0''))\cdot\hat{\mathbf{z}} \qquad \text{(B.C. 2)}$$

$$\mathbf{E}_0'\times\hat{\mathbf{z}}=(\mathbf{E}_0+\mathbf{E}_0'')\times\hat{\mathbf{z}} \qquad \text{(B.C. 3)}$$

$$\frac{1}{\mu'}(\mathbf{k}'\times\mathbf{E}_0')\times\hat{\mathbf{z}}=\left(\frac{1}{\mu}(\mathbf{k}\times\mathbf{E}_0)+\frac{1}{\mu}(\mathbf{k}''^*\times\mathbf{E}_0'')\right)\times\hat{\mathbf{z}} \qquad \text{(B.C. 4)}$$

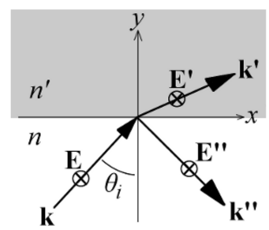
- As mentioned before, the polarization vectors $\mathbf{E}_0$, $\mathbf{E}_0'$, and $\mathbf{E}_0''$ are complex numbers and have two components. It is simplest to think of a general polarization as the sum of two linearly-polarized waves that are orthogonal. Let us solve this problem for the two special cases of linear polarization: perpendicular to the plane of incidence and parallel to the plane of incidence. We can always find other solutions through superposition.
- For electric fields always perpendicular to the plane of incidence, the boundary conditions reduce to:

$$0=0 \qquad \text{(B.C. 1)}$$

$$\sqrt{\mu'\epsilon'}\sin\theta_t E_0'=\sqrt{\mu\,\epsilon}\sin\theta_i E_0+(\sqrt{\mu\,\epsilon})^*\sin\theta_i E_0'' \qquad \text{(B.C. 2)}$$

$$E_0'=E_0+E_0'' \qquad \text{(B.C. 3)}$$

$$\sqrt{\epsilon'/\mu'}\,E_0'\cos\theta_t=\sqrt{\epsilon/\mu}\cos\theta_i E_0-(\sqrt{\epsilon/\mu})^*\cos\theta_i E_0'' \qquad \text{(B.C. 4)}$$




- Note that we have assumed that the permeability is real-valued, as is approximately true in most cases, so that $\mu^* = \mu$. Therefore, a complex-valued $\mathbf{k}''$ is caused by a complex-valued $\epsilon$ (or a negative $\epsilon$ since the square root of a negative number is an imaginary number).
- It should be noted that when we took the cross product with the *z* axis in the second and fourth boundary conditions, the answer was a vector equation. We then matched up components to get the final scalar equation.
- The first boundary condition is not helpful in this polarization.
- The second boundary condition when combined with the third is just a restatement of Snell's law.
- The third and fourth boundary conditions can be combined to solve for the fields in terms of the incident field strength.

$$E_0'=\frac{\sqrt{\epsilon/\mu}\cos\theta_i+(\sqrt{\epsilon/\mu})^*\cos\theta_i}{\sqrt{\epsilon'/\mu'}\cos\theta_t+(\sqrt{\epsilon/\mu})^*\cos\theta_i}E_0$$

$$E_0''=\frac{\sqrt{\epsilon/\mu}\cos\theta_i-\sqrt{\epsilon'/\mu'}\cos\theta_t}{(\sqrt{\epsilon/\mu})^*\cos\theta_i+\sqrt{\epsilon'/\mu'}\cos\theta_t}E_0$$

- We use Snell's Law to transform the unknown transmitted angle into the known incidence angle. We also rewrite each permittivity in terms of index of refraction:

$$E_0' = \frac{2\Re(n)\cos\theta_i}{n^*\cos\theta_i + n'\frac{\mu}{\mu'}\sqrt{1-(n/n')^2\sin^2\theta_i}} E_0$$

$$E_0'' = \frac{n\cos\theta_i - n'\frac{\mu}{\mu'}\sqrt{1-(n/n')^2\sin^2\theta_i}}{n^*\cos\theta_i + n'\frac{\mu}{\mu'}\sqrt{1-(n/n')^2\sin^2\theta_i}} E_0$$

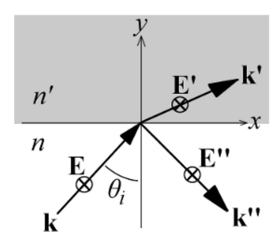
- Of considerable interest is the reflection coefficient $R$ which measures the fraction of the incident power that is reflected, and the transmission coefficient $T$ which measures the fraction of the incident power that is transmitted.

$$R = \left|\frac{E_0''}{E_0}\right|^2 \quad \text{and} \quad T = \frac{\Re(n')}{\Re(n)} \frac{\mu\sqrt{1-(n/n')^2\sin^2\theta_i}}{\mu'\cos\theta_i} \left|\frac{E_0'}{E_0}\right|^2$$

- The derivation of the equations above is shown in the Appendix at the end of this document.
- Use of these definitions for $R$ and $T$ leads to what are known as the "Fresnel Equations". Note that often the materials are assumed to be non-magnetic so that the $\mu/\mu'$ factors are omitted.

*Fresnel Equations: Polarization Perpendicular to the Plane of Incidence*

$$R = \left|\frac{n\cos\theta_i - n'\frac{\mu}{\mu'}\sqrt{1-(n/n')^2\sin^2\theta_i}}{n^*\cos\theta_i + n'\frac{\mu}{\mu'}\sqrt{1-(n/n')^2\sin^2\theta_i}}\right|^2$$

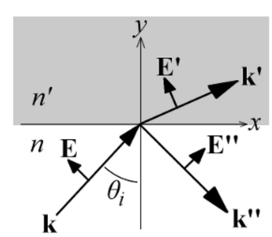



$$T = \frac{\Re(n')}{\Re(n)} \frac{\mu\sqrt{1-(n/n')^2\sin^2\theta_i}}{\mu'\cos\theta_i} \left|\frac{2\Re(n)\cos\theta_i}{n^*\cos\theta_i + n'\frac{\mu}{\mu'}\sqrt{1-(n/n')^2\sin^2\theta_i}}\right|^2$$

- It is left as an exercise for the interested student to derive the reflection and transmission coefficients for the case of electric field vectors parallel to the plane of incidence. The results are:

*Fresnel Equations: Polarization Parallel to the Plane of Incidence*

$$R = \left|\frac{n'\cos\theta_i - n\frac{\mu'}{\mu}\sqrt{1-(n/n')^2\sin^2\theta_i}}{n'\cos\theta_i + n^*\frac{\mu'}{\mu}\sqrt{1-(n/n')^2\sin^2\theta_i}}\right|^2$$




$$T = \frac{\Re(n')}{\Re(n)} \frac{\mu\sqrt{1-(n/n')^2\sin^2\theta_i}}{\mu'\cos\theta_i} \left|\frac{2\Re(n)\cos\theta_i}{n'\frac{\mu}{\mu'}\cos\theta_i + n^*\sqrt{1-(n/n')^2\sin^2\theta_i}}\right|^2$$

- Note that because of conservation of energy, $R + T = 1$. In order for the conservation of energy to always hold, even for lossy initial material and lossy final material, these exact forms must be used instead of the approximate forms commonly shown in textbooks.

## 8. Brewster's Angle

- For non-magnetic material ($\mu/\mu' = 1$), is there an angle of incidence for which there is no reflected wave (i.e. for which $R = 0$) and therefore the wave energy is totally transmitted?
- For polarization perpendicular to the plane of incidence, we set $R = 0$ and try to solve for the angle of incidence:

$$0 = \left| \frac{n \cos\theta_i - n'\sqrt{1-(n/n')^2 \sin^2\theta_i}}{n^* \cos\theta_i + n'\sqrt{1-(n/n')^2 \sin^2\theta_i}} \right|^2$$

$$n^2 \cos^2\theta_i = n'^2 - n^2 \sin^2\theta_i$$

$$n^2 = n'^2$$

- Therefore, for perpendicular polarization there is zero reflection only if there is no difference in material, which is the trivial case.
- For polarization parallel to the plane of incidence, we set $R = 0$ and try to solve for the angle of incidence:

$$0 = \left| \frac{n' \cos\theta_i - n\sqrt{1-(n/n')^2 \sin^2\theta_i}}{n' \cos\theta_i + n^*\sqrt{1-(n/n')^2 \sin^2\theta_i}} \right|^2$$

$$0 = n' \cos\theta_i - n\sqrt{1-(n/n')^2 \sin^2\theta_i}$$

$$(n'^2 - n^2)\cos^2\theta_i = (n^2 - n^4/n'^2)\sin^2\theta_i$$

$$(n'^2 - n^2) = \frac{n^2}{n'^2}(n'^2 - n^2)\tan^2\theta_i$$

$$\boxed{\theta_i = \theta_B = \tan^{-1}\left(\frac{n'}{n}\right)}$$ *Brewster's Angle*

- This is known as Brewster's angle. At this angle of incidence, parallel-polarized waves are completely transmitted.
- As the refractive index of the material being penetrated $n'$ becomes much greater than the outside material $n$, the Brewster's angle asymptotically approaches ninety degrees.
- If an unpolarized or mixed-polarization wave is incident at the Brewster's angle, only waves perpendicular to the plane of incidence get reflected back, and the rest of the wave is transmitted through. This setup can be used as a polarizer.
- For common material combinations such as air-to-water and air-to-glass, Brewster's angle is between 50 and 60 degrees.
- What we experience as glare from the sun is typically light waves that are strongly polarized

perpendicular to the plane of incidence. When the sun is low, its sunlight strikes flat water, flat snow, or flat wet streets near Brewster's angle so that only perpendicular-polarized light is reflected into your eye as glare. Certain sunglasses take advantage of this fact to block the glare. Such sunglasses contain a polarization filter that only lets in light polarized parallel to the plane of incidence.

- The physics of Brewster's angle is quite simple. The electric charge dipoles in the material oscillate in response to the electric fields of the wave, and thus oscillate in the same direction as the electric field vectors. Propagation of a wave in a dielectric material can be thought of as a continual absorption and re-radiation of the wave by the oscillating electric dipoles of the material. Oscillating dipoles do not radiate in the direction of oscillation. The Brewster's angle is therefore the configuration where the electric field vector in the penetrated material, and thus the oscillation direction, points in the same direction as the potential reflected wave direction of propagation. In other words, this occurs when the transmitted wavevector is perpendicular to the reflected wavevector.
- In fact we can use this concept to re-derive Brewster's Law:

$$\theta_i + \theta_t = 90° \quad \text{(remember that the angles are defined relative to the } z \text{ axis)}$$

$$\sin(\theta_t) = \sin(90° - \theta_i)$$

$$\sin(\theta_t) = \cos(\theta_i)$$

- Using Snell's Law:

$$\frac{n}{n'}\sin(\theta_i) = \cos(\theta_i)$$

$$\theta_i = \theta_B = \tan^{-1}\left(\frac{n'}{n}\right)$$

## 9. Total Internal Reflection

- Let us now ask the opposite question. Is there an angle of incidence where the wave is completely reflected? Let us set $R = 1$ and solve for the angle:

$$1 = \left| \frac{n\cos\theta_i - n'\sqrt{1-(n/n')^2\sin^2\theta_i}}{n*\cos\theta_i + n'\sqrt{1-(n/n')^2\sin^2\theta_i}} \right|^2$$

$$\frac{\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i} = \frac{-\mu}{\mu'}\sqrt{n'^2 - n^2\sin^2\theta_i}$$

$$n'^2 - n^2\sin^2\theta_i = 0$$

$$\boxed{\theta_i = \theta_C = \sin^{-1}\left(\frac{n'}{n}\right)} \qquad \textit{Total Internal Reflection Critical Angle}$$

- We note that only when $n' < n$ do we end up with a valid angle. That is why the reflection is

called “internal”. It only happens when attempting to leave a high index material such as glass or water and enter a low-index material such as air.
- At this angle, the transmitted wave is refracted so much that it essentially propagates along the boundary and never penetrates the material. We could even use this concept to re-derive the equation for total internal reflection:

$$\theta_t = 90°$$

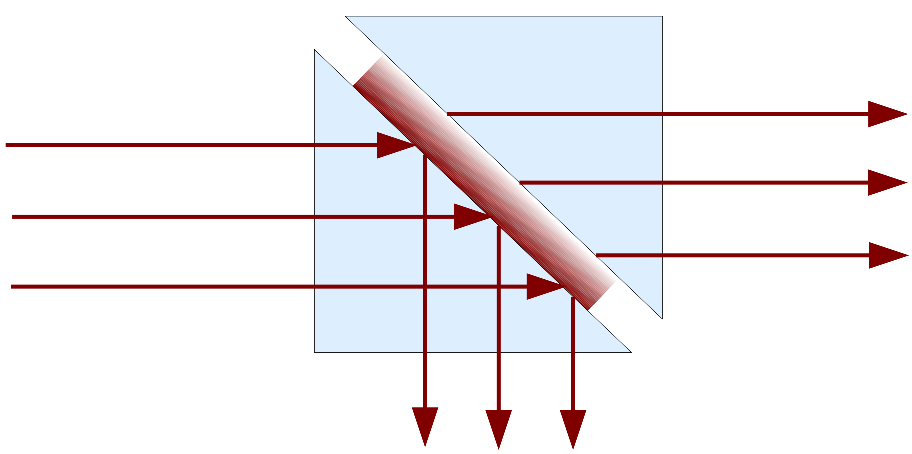
$$\sin(\theta_t) = 1$$

- Using Snell's Law:

$$\frac{n}{n'} \sin(\theta_i) = 1$$

$$\theta_i = \sin^{-1}\left(\frac{n'}{n}\right)$$

- For angles of incidence greater than this critical angle, there is still no transmission. The transmitted wave becomes a wave along the boundary known as an “evanescent wave”.
- The evanescent wave penetrates slightly into the second region but dies off exponentially in this direction, and carries no energy into the second region.
- If another piece of high-index material is placed near the first piece, but a gap remains, the waves can tunnel through the gap through the mechanism of the evanescent waves. This is a classic example of wave tunneling. This process is called “frustrated total internal reflection”.
- Note that to accurately describe frustrated total internal reflection mathematically, we have to start over from Maxwell's equations and can't use the Fresnel equations because there are resonances in the gap.
- This is the principal mechanism behind a beam-splitter cube. The transmission coefficient can be controlled by varying the spacing of the prisms.



## Appendix

The reflection coefficient *R* and transmission coefficient *T* measure that fraction of the total incident power that is reflected and transmitted, respectively. Since they measure electromagnetic power propagation (which is described by the Poynting vector), they are proportional to the magnitude squared of the electric field. However, there are also other constants that can't be neglected which account for the power getting spread out over a larger area due to refraction. Let us derive the proper expressions.

A plane, linearly polarized, monochromatic electromagnetic wave traveling in the **k** direction in a linear, complex-valued medium has wave number $k=\beta+i\alpha/2$ where $k=\omega\sqrt{\mu\epsilon}$ , and has an electric field:

$$\mathbf{E}=E_0\boldsymbol{\epsilon}_1 e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{x}/2}e^{i\beta\hat{\mathbf{k}}\cdot\mathbf{x}-i\omega t+i\theta}$$

Here, $E_0$ is the real-valued magnitude of the electric field at $z = 0$ and time $t = \theta/\omega$, $\boldsymbol{\epsilon}_1$ is the real-valued polarization unit vector, $\alpha$ is twice the imaginary part of the wave number or attenuation constant, $\beta$ is the real part of the wave number, $\theta$ is the constant phase of the wave, and $\omega$ is the temporal angular frequency.

Inserting this into Faraday's law leads directly to the associated magnetic field

$$\mathbf{B}=\frac{n}{c}E_0(\hat{\mathbf{k}}\times\boldsymbol{\epsilon}_1)e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{x}/2}e^{i\beta\hat{\mathbf{k}}\cdot\mathbf{x}-i\omega t+i\theta}$$ where the index of refraction is complex: $n=\sqrt{\epsilon\mu/\epsilon_0\mu_0}$

The time-averaged complex-valued power density transmitted by the wave in the **k** direction is the complex Poynting vector:

$$\hat{\mathbf{k}}\cdot\mathbf{S}=\frac{1}{2\mu}\hat{\mathbf{k}}\cdot\mathbf{E}\times\mathbf{B}^*$$

Inserting the electric and magnetic field expressions above, we find:

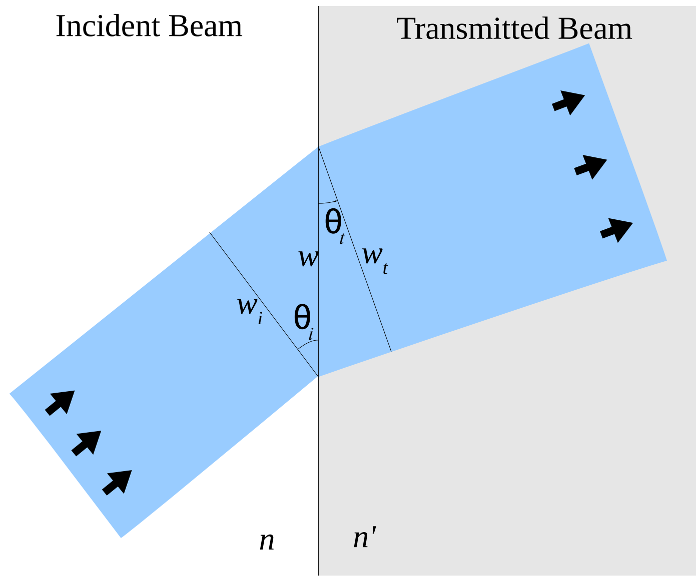
$$\hat{\mathbf{k}}\cdot\mathbf{S}=\frac{n^*}{2\mu c}e^{-\alpha\hat{\mathbf{k}}\cdot\mathbf{x}}|E_0|^2$$

The fraction of incident power *T* transmitted through a surface is the total power of the wave at the surface on the transmitted side divided by the total incident power at the surface on the incident side:

$$T=\frac{\Re(\hat{\mathbf{k}}\cdot\mathbf{S}_t(\mathbf{x}=0))}{\Re(\hat{\mathbf{k}}\cdot\mathbf{S}_i(\mathbf{x}=0))}\frac{A_t}{A_i}$$

Even though we are dealing with plane waves, for the purpose of calculating the areas, pretend we have square beams. The depth of the incident and transmitted beam will be the same, but the widths will be different because refraction causes the transmitted beam to spread out. In order to calculate the ratio of widths, we must relate the two widths. Let us relate them both to the projected width *w*, which is the same for both, as shown in the figure: $w_i=w\cos\theta_i$ and $w_t=w\cos\theta_t$ . Then:

$$\frac{A_t}{A_i}=\frac{l\,w_t}{l\,w_i}=\frac{l\,w\cos\theta_t}{l\,w\cos\theta_i}=\frac{\cos\theta_t}{\cos\theta_i}$$




Use Snell's law to get rid of the transmitted angle:

$$\frac{A_t}{A_i}=\frac{\sqrt{1-(n/n')^2\sin^2\theta_i}}{\cos\theta_i}$$

Using this ratio, the transmission becomes:

$$T=\frac{\Re(\hat{\mathbf{k}}\cdot\mathbf{S}_t(\mathbf{x}=0))}{\Re(\hat{\mathbf{k}}\cdot\mathbf{S}_i(\mathbf{x}=0))}\frac{\sqrt{1-(n/n')^2\sin^2\theta_i}}{\cos\theta_i}$$

Insert the power densities to find:

$$\boxed{T=\frac{\Re(n')}{\Re(n)}\frac{\mu\sqrt{1-(n/n')^2\sin^2\theta_i}}{\mu'\cos\theta_i}\left|\frac{E_0'}{E_0}\right|^2}$$

Similarly, we find the fraction of power reflected:

$$R=\frac{P_r(\mathbf{x}=0)}{P_i(\mathbf{x}=0)}$$

$$R=\frac{\Re(\hat{\mathbf{k}}\cdot\mathbf{S}_r(\mathbf{x}=0))}{\Re(\hat{\mathbf{k}}\cdot\mathbf{S}_i(\mathbf{x}=0))}\frac{A_i}{A_i}$$

$$\boxed{R=\left|\frac{E''_0}{E_0}\right|^2}$$