# The Uniqueness of Maxwell's Equations

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## 1. Introduction

The question is often asked, “Why do Maxwell's equations contain eight scalar equations if there are only six unknowns? Aren't some of the equations redundant, and if not, isn't the problem over-specified?” This paper attempts to answer this question. The short answer is that Maxwell's equations are neither redundant nor over-specified because only six of Maxwell's equations are dynamical. The other two can be thought of as initial conditions. Note that although not typically written down explicitly as part of Maxwell's equations, boundary conditions are also considered part of the system. Maxwell's equations in their complete form involve six linear partial differential equations, six unknowns, initial conditions and boundary conditions and therefore they have a unique solution according to traditional theorems of linear algebra.

## 2. Definitions

Let us first make some definitions and dispel common misconceptions. The following analysis focuses on Maxwell's equations in vacuum. Including the effects of materials would complicate the analysis without changing the core arguments. In modern vector notation and in SI units, Maxwell's equations in vacuum are:

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0} \qquad \nabla\cdot\mathbf{B}=0$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B}=\mu_0\mathbf{J}+\mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

*Maxwell's equations in vacuum in SI units*

Here, **E** is the total electric field, **B** is the total magnetic field, $\rho$ is the electric charge density, **J** is the electric current density, $\epsilon_0$ is the permittivity in free space, $\mu_0$ is the permeability of free space, $\nabla\cdot()$ is the divergence operator, and $\nabla\times()$ is the curl operator. We omit hypothetical magnetic charges and magnetic currents as they have little bearing on the known universe and only unnecessarily complicate the analysis without changing the core arguments.

The six unknowns to be found are the components of the electromagnetic field:

$E_x, E_y, E_z, B_x, B_y, B_z$ *The six unknowns (dependent variables) of Maxwell's equations*

They are not numbers to be found as in linear algebra, rather they are functions of $x$, $y$, $z$, and $t$ that need to be determined. In the language of differential calculus, these six unknowns are dependent variables which depend on the four independent variables:

$x, y, z, t$ *The independent variables of Maxwell's equations*

When solving differential equations, we consider a solution to be unique when there is one and only one functional form that we can write down for each dependent variable and this functional form includes only known constants and independent variables, but no dependent variables or derivatives. The charge density and current density are not unknowns in this modern condensed form. If they were, then everything would be unknown and there would be nothing to solve. Charges and currents create fields; the charges and currents are the sources. (Historically, Maxwell's original work treated the charges and currents as unknowns, but he also included extra equations which effectively turned them into knowns.)

The beginner student may look at Maxwell's equations and think there are only *four* equations and six unknowns, and therefore the problem is underspecified. From a physical standpoint, Maxwell's equations are four equations constituting four separate laws: Coulomb's law, the Maxwell-Ampere law, Faraday's law, and the no-magnetic-charge law. But from a mathematical standpoint, there are eight equations because two of the physical laws are vector equations with multiple components. In component form in rectangular coordinates, the full eight equations are:

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}=\frac{\rho}{\epsilon_0} \qquad \frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0$$

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}=-\frac{\partial B_x}{\partial t} \qquad \frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}=\mu_0 J_x+\mu_0\epsilon_0\frac{\partial E_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}=-\frac{\partial B_y}{\partial t} \qquad \frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}=\mu_0 J_y+\mu_0\epsilon_0\frac{\partial E_y}{\partial t}$$

$$\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=-\frac{\partial B_z}{\partial t} \qquad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=\mu_0 J_z+\mu_0\epsilon_0\frac{\partial E_z}{\partial t}$$

*Maxwell's equations in component form*

We therefore seem to have eight independent equations in six unknowns and the problem seems to be over-specified (or redundant) according to linear algebra. The eager student may quickly reply that perhaps Maxwell's equations are non-linear. But a perusal of the above equations reveals that each term involves a single dependent variable raised to the power of one, and therefore the system is linear. In theory, because the system is linear, we can decouple all of the equations and end up with each equation containing only one dependent variable. This is in fact what happens when Maxwell's equations are put in wave-equation form, as is done later.

In differential calculus, two distinct layers of information must be present in order to have a completely unique solution: (1) sufficient differential equations to determine the dependent variables, and (2) sufficient boundary conditions to determine the integration constants. The term “boundary conditions” used here includes initial conditions, as initial conditions can be thought of as conditions on the boundary of time. If the system is linear, as is true for Maxwell's equations, the differential equations are sufficient for a unique solution if the number of equations equals the number of unknowns, and if all the equations are linearly dependent. The boundary conditions are sufficient for a unique solution if the number of known boundary conditions equals the number of dependent variables times the number of independent variables times the order of the differential equations, and if the boundary conditions are not ill-posed mathematically. This number can be understood from the fact that every time we integrate away a derivative operator, we introduce an integration constant that must be determined using boundary conditions.

In Maxwell's equations, we have first-order differential equations, six dependent variables, and four independent variables. We therefore need 24 boundary conditions for a unique solution. Note that in most physics problems, it does not seem like we have 24 boundary conditions because of the presence of symmetries or trivial boundary conditions. For instance, if we are doing an electrostatic problem, 12 of the boundary conditions are trivially zero because there are no magnetic fields present.

The issue of sufficient boundary conditions is a concern only during the *application* of Maxwell's equations to a specific situation, but does not concern the Maxwell's equations themselves. For the purposes of our analysis, we assume that the student knows how to properly construct and apply boundary conditions to arrive at a unique solution. We can therefore safely ignore boundary condition considerations from here on and focus on the other layer: the presence of a sufficient number of differential equations to determine the dependent variables. If Maxwell's equations are sufficient for a unique solution but not over-specified, we would expect six equations in six unknowns, plus appropriate boundary/initial conditions.

## 3. The Uniqueness of Maxwell's Equations in Standard Form

According to the Helmholtz decomposition theorem (the fundamental theorem of vector calculus) every well-behaved vector field **A** can be decomposed into a sum of a transverse vector field and a longitudinal vector field:

$$\mathbf{A}=\mathbf{A}_t+\mathbf{A}_l \text{ where } \nabla\times\mathbf{A}=\nabla\times\mathbf{A}_t \text{ (so that } \nabla\times\mathbf{A}_l=0\text{) and } \nabla\cdot\mathbf{A}=\nabla\cdot\mathbf{A}_l \text{ (so that } \nabla\cdot\mathbf{A}_t=0\text{)}$$

The transverse part $\mathbf{A}_t$ is a curling (i.e. solenoidal, rotational, non-diverging) vector field. The longitudinal part is $\mathbf{A}_l$ is a diverging (i.e. irrotational, non-curling) vector field. The Helmholtz decomposition theorem arises from the fact that the divergence operator and the curl operator can be thought of as orthogonal operators. This is because the curl of the gradient is always zero, $\nabla\times\nabla\Phi=0$ and the divergence of the curl is always zero, $\nabla\cdot(\nabla\times\mathbf{D})=0$ . Note that the terms “longitudinal” and “transverse” refer to the directionality of the operators, and not necessarily the directionality of the vectors. In other words, $\mathbf{A}_t$ is the component that results when the transverse differential (the curl) is taken, it is not the component that is always transverse to some reference vector.

Note that there is another part to the vector field that is both non-curling and non-diverging. If a vector field has zero divergence and zero curl, it can still have something else left called the relaxed part (or the Laplacian part). It is called the relaxed part because charges and currents are what create diverging and curling fields, so in their absence the fields relax to a state that minimizes potential energy while still meeting all boundary conditions. The relaxation method is a common numerical method for finding the relaxed state of any vector field. We can show the relaxed nature mathematically. If any vector field **A** is non-curling, $\nabla\times\mathbf{A}=0$ , then because of the mathematical identity $\nabla\times\nabla\Phi=0$ we must have $\mathbf{A}=-\nabla\Phi$ . If the vector field is also non-diverging, $\nabla\cdot\mathbf{A}=0$ , then upon inserting the negative gradient of the scalar field, we find $\nabla^2\Phi=0$ . This is the Laplace equation and the solutions to this equation are the relaxed part of **A**. Imagine stretching a rubber sheet and fixing it to an irregularly shaped rim. It's final shape is a minimal surface analogous to the relaxed state of a vector field. Thus we see that even if a vector field is non-curling and non-diverging, it can still have a non-zero and non-trivial functionality. The relaxed part of a vector field is contained in $\mathbf{A}_t$ and $\mathbf{A}_l$ as should be obvious from the above analysis. The relaxed part is determined solely by boundary conditions. For this reason, we can assume boundary conditions are properly applied so that the relaxed parts are uniquely determined without going into any more detail.

We can expand the electric and magnetic fields in Maxwell's equations into their longitudinal and transverse components in order to attempt to better analyze the role of each equation and whether there is redundancy. We expand the fields according to:

$$\mathbf{E}=\mathbf{E}_t+\mathbf{E}_l \quad \text{and} \quad \mathbf{B}=\mathbf{B}_t+\mathbf{B}_l$$

Inserting these expansions into Maxwell's equations and dropping terms that are identically zero (such as $\nabla\cdot\mathbf{E}_t=0$ ), they become:

(1) $\nabla\cdot\mathbf{E}_l=\frac{\rho}{\epsilon_0}$ (2) $\nabla\cdot\mathbf{B}_l=0$

(3) $\nabla\times\mathbf{E}_t=-\frac{\partial\mathbf{B}_t}{\partial t}-\frac{\partial\mathbf{B}_l}{\partial t}$ (4) $\nabla\times\mathbf{B}_t=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial\mathbf{E}_t}{\partial t}+\frac{1}{c^2}\frac{\partial\mathbf{E}_l}{\partial t}$

Taking the divergence of equations (3) and (4) and using the continuity equation, $\nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}$ ,we arrive at the modified forms of these equations:

(3') $\frac{\partial}{\partial t}(\nabla\cdot\mathbf{B}_l)=0$ (4') $\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}_l)=\frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right)$

Upon comparing equations (3') and (4') to equations (2) and (1) respectively, they may seem to be identical. In other words, equations (1) and (2) of Maxwell's equations seem to be redundant because they are already contained in equations (3) and (4) as made clear in their modified form shown in equations (3') and (4'). Equations (3) and (4), which are Faraday's law and the Maxwell-Ampere law, constitute six equations in six unknowns, so it makes sense that equations (1) and (2) - Coulomb's law and the no-magnetic-charge law - could be redundant. So why are Coulomb's law and the no-magnetic-charge law always included as part of Maxwell's equations?

The answer is that they are not redundant, and the reason why is because equation (3') does not exactly match equation (2) and equation (4') does not exactly match equation (1). The presence of the time derivative makes all the difference. Equations (3') and (4') do not tell us the longitudinal components of the electric and magnetic fields, they only tell as the *time-evolution* of the longitudinal components of the fields. We still need equations (1) and (2) in order to find the *initial* longitudinal components. So Coulomb's law and the no-magnetic-charge law are not redundant.

The subtle effect of the time-derivatives contained in equations (3') and (4') can be made clearer by integrating them away. We have to be careful because when we integrate, we have to remember to include the integration constant. But because it is a partial derivative, the integration constant is not a pure constant, it is only constant with respect to time. It could still be a function of the other dependent variables. Upon integrating equations (3') and (4'), we find that Faraday's law and the Ampere-Maxwell law only contain the information:

(3'') $\nabla\cdot\mathbf{B}_l=f(x,y,z)$ (4'') $\nabla\cdot\mathbf{E}_l=\frac{\rho}{\epsilon_0}+g(x,y,z)$

where *f* and *g* are unknown functions. Comparing equations (3'') and (4'') to equations (1) and (2) we see that there is no redundancy after all. We need Coulomb's law and the no-magnetic-charge law in order to determine the functions *f* and g (even though they end up being zero). The fact that general conceptual arguments can tell us that *f* and *g* above are zero, or the fact that the *first two* of Maxwell's equations tell us that *f* and *g* are zero may confuse some people into thinking that the other Maxwell equations, (3) and (4), tell us that *f* and *g* are zero. This would imply redundancy. But from a mathematical perspective, Faraday's law and the Ampere-Maxwell law do not uniquely specify the divergence of the fields, and thus there is no redundancy.

If Maxwell's equations are not redundant, then they seem to be over-specified because we still have eight equations in six unknowns. But Maxwell's equations are not over-specified and the reason is because equations (1) and (2) do not really count as part of the system of linear equations – they count only as initial conditions. They *are* needed to uniquely determine a solution, but they are needed only as initial conditions and not as part of the system of linear independent differential equations. Once (1) and (2) are used to find the initial state of the longitudinal components of the fields, then equations (3) and (4) dictate the time evolution of the longitudinal components at all future times, as made explicit in equations (3') and (4').

The time evolution of the longitudinal components turns out being statically linked. (That is, the divergence of the longitudinal electric field is linked to the charge density at all times *in the same way* it was initially linked. The relationship is static, but $\mathbf{E}_l$ itself is not static. The quantity $\nabla\cdot\mathbf{E}_l$ instantaneously tracks the charge density. Some books calls this pseudo-static.) But the static behavior does not change the mathematical arguments that equations (3) and (4) are a complete description of the dependent variables, including the dynamical evolution of the longitudinal components, and equations (1) and (2) are merely initial conditions. Because the dynamical behavior of the longitudinal components is static, the *initial values* for the longitudinal components end up being the *same values* through all time. As a result, Coulomb's law (1) and the no-magnetic-charge law (2) end up being valid for all time and not just at an initial time. But this is just a quirk of the physics because their dynamical evolution is static, and is not a mathematical paradox. Perhaps for this reason, Coulomb's law and the no-magnetic-charge law are often incorrectly elevated to be considered part of the system of differential equations in a linear-algebra sense, when they should only be regarded as boundary conditions in time.

In summary, Maxwell's equations are neither over-specified (six equations in six unknowns), nor are they redundant (the divergence equations are needed for a unique solution) when we recognize Coulomb's law and the no-magnetic-charge law as boundary conditions in time.

Strictly speaking, equation (2) does not fully specify $\mathbf{B}_l$, it only specifies the diverging part. There is still a relaxed part that is determined by boundary conditions. This means that equations (3') and (2) do not doom $\mathbf{B}_l$ to be initially and forevermore zero. They only doom the $\nabla\cdot\mathbf{B}_l$ to be once and forevermore zero. The relaxed part of $\mathbf{B}_l$ can still be non-zero and can even change in time. For instance, the magnetic field inside an ideal, infinite solenoid is non-curling and non-diverging, but is still real and non-zero, and can even change in time as we change the current in the solenoid. This subtlety is partly what keeps causality from being violated.

Consider if at some time $t_0$ we "turn on" a point electric charge $q$ that was not there before and leave it on. Static point charges create the pseudo-static (i.e. instantaneous) divergence of the longitudinal electric field $\mathbf{E}_l$ according to (1) and (4'). I might therefore conclude that at the exact moment I turn on $q$, a man on the moon can detect its longitudinal field. This would clearly violate causality. The error in our reasoning is that we assumed $\mathbf{E}_l$ is instantaneous, when clearly equation (4') only specifies that $\nabla\cdot\mathbf{E}_l$ is statically and therefore instantaneously linked to the charge density. The quantity $\nabla\cdot\mathbf{E}_l$ is not an independent physical quantity we can measure. We can only measure and give physical reality to the total field **E**. Therefore causality is not violated. If we worked out the mathematics of a point charge turning on, we would find that there are terms in the dynamical equations that cancel the seeming instantaneous fields beyond the causality shell. For a good exposition of how causality is not violated despite Coulomb's law seeming to be instantaneous, see J. D. Jackson, Eur. J. Phys. 31 L79 (2010).

## 4. Uniqueness of the Wave Form of Maxwell's Equations

We can cast Maxwell's equations into a wave form. Take Ampere's law and take the partial derivative with respect to time on both sides:

$$\nabla \times \left( \frac{\partial \mathbf{B}}{\partial t} \right) = \mu_0 \frac{\partial \mathbf{J}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Faraday's Law specifies the partial of **B** with respect to *t*, so we can insert it into this equation to find:

$$\nabla \times (\nabla \times \mathbf{E}) = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$ , this becomes:

$$\nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

The divergence of **E** is specified by Coulomb's law, so we can insert it in to find:

$$\boxed{\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \frac{1}{\epsilon_0} \nabla \rho + \mu_0 \frac{\partial \mathbf{J}}{\partial t}}$$

This is a differential equation involving only **E** and known sources. The electric field has been mathematically decoupled from the magnetic field. Because we inserted Coulomb's law, we may be tempted to say that this equation contains Coulomb's law and therefore Coulomb's law by itself has become redundant. In fact, based on the way we inserted it, this equation only contains the *gradient* of Coulomb's law. We therefore still need Coulomb's law for a complete solution.

We can do the same thing for the magnetic field. Start with Faraday's law and take the partial derivative with respect to time on both sides:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \left( \frac{\partial \mathbf{E}}{\partial t} \right) = -\frac{\partial^2 \mathbf{B}}{\partial t^2}$$

The partial of **E** is found in the Ampere-Maxwell law, so we can insert it into this equation to find:

$$\nabla\times\nabla\times\mathbf{B}=\mu_0\nabla\times\mathbf{J}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

Again, using the vector identity $\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$, this becomes:

$$\nabla(\nabla\cdot\mathbf{B})-\nabla^2\mathbf{B}=\mu_0\nabla\times\mathbf{J}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}$$

The divergence of **B** is specified to be zero by the no-magnetic-charge law, so that we end up with:

$$\boxed{\nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=-\mu_0\nabla\times\mathbf{J}}$$

This is a differential equation involving only **B** and known sources. The magnetic field has been mathematically decoupled from the electric field. Because we inserted the no-magnetic-charge law, we may be tempted to say that this equation contains that law and therefore the no-magnetic-charge law has become redundant. In fact, based on the way we inserted it, this equation only contains the *gradient* of the no-magnetic-charge law. We therefore still need the no-magnetic-charge law for a complete solution.

In summary, Maxwell's equations in wave-equation form are:

$$\nabla\cdot\mathbf{E}=\frac{\rho}{\epsilon_0} \qquad \nabla\cdot\mathbf{B}=0$$

$$\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=\frac{1}{\epsilon_0}\nabla\rho+\mu_0\frac{\partial\mathbf{J}}{\partial t} \qquad \nabla^2\mathbf{B}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2}=-\mu_0\nabla\times\mathbf{J}$$

*Maxwell's equations in wave-equation form*

We have to include the first two equations to get a unique solution for the same reason as in the original form. Again, the first two equations serve as initial conditions and the second two represent six linear differential equations in six unknowns. This form is fully equivalent to the original form. Changing Maxwell's equations to this form does not reduce the number of unknowns and does not reduce the number of physically relevant equations. What it does accomplish is it completely decouples all of the dependent variables. Each of the six dynamical differential equations now contains one and only one dependent variable (a field component). The *decoupling* is what makes the wave-equation form of Maxwell's equations so desirable, not the fact that we have reduced the number of unknowns or the number of equations.

Expanding the fields in Maxwell's equations into longitudinal and transverse components exactly as we did previously, we find:

(5) $\nabla\cdot\mathbf{E}_l=\frac{\rho}{\epsilon_0}$ (6) $\nabla\cdot\mathbf{B}_l=0$

(7) $\nabla(\nabla\cdot\mathbf{E}_l)-\nabla\times(\nabla\times\mathbf{E}_t)-\frac{1}{c^2}\frac{\partial^2\mathbf{E}_l}{\partial t^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}_t}{\partial t^2}=\frac{1}{\epsilon_0}\nabla\rho+\mu_0\frac{\partial\mathbf{J}}{\partial t}$

(8) $\nabla(\nabla\cdot\mathbf{B}_l)-\nabla\times(\nabla\times\mathbf{B}_t)-\frac{1}{c^2}\frac{\partial^2\mathbf{B}_l}{\partial t^2}-\frac{1}{c^2}\frac{\partial^2\mathbf{B}_t}{\partial t^2}=-\mu_0\nabla\times\mathbf{J}$

Take the divergence of the last two equations and, using the continuity equation, $\nabla\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}$ , we find modified forms of these equations:

(7') $(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})(\nabla\cdot\mathbf{E}_l-\frac{\rho}{\epsilon_0})=0$ (8') $(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2})(\nabla\cdot\mathbf{B}_l)=0$

Comparing equations (7') and (8') to equations (5) and (6), we again see that the dynamical equations (7) and (8) seem to contain the divergence equations, but they in fact only contain derivatives of the divergence equations. Even though equations (5) and (6) are only initial conditions, we still need them in order to find a unique solution. We can see this by integrating away the wave-operator:

(7'') $\nabla\cdot\mathbf{E}_l=\frac{\rho}{\epsilon_0}+\int P(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}d^3\mathbf{k}$ (8'') $\nabla\cdot\mathbf{B}_l=\int Q(\mathbf{k})e^{i\mathbf{k}\cdot\mathbf{x}-ickt}d^3\mathbf{k}$

Comparing equations (7'') and (8'') to equations (5) and (6), we see that we need equations (5) and (6), the initial conditions, to determine that there are no extra terms in (7'') and (8'')

## 5. Uniqueness of the Potentials Form of Maxwell's Equations

Another form of Maxwell's equations can be found by defining potentials. Instead of using some prior knowledge of Maxwell's equation to take shortcuts as is traditionally done, let us start as general as possible and let the facts fall out along the way so that we can keep track of the number of unknowns and the number of independent dynamical differential equations. The magnetic field and electric field have curling parts and diverging parts. We can also explicitly write out a time-derivative part in order to match traditional potential definitions even though Helmholtz's theorem does not require it. The time-derivative term may also have a curling or diverging nature. These parts are defined in terms of potentials:

$$\mathbf{B}=\nabla\times\mathbf{A}_B-\nabla\Phi_B-\frac{\partial\mathbf{A}'_B}{\partial t} \quad \text{and} \quad \mathbf{E}=\nabla\times\mathbf{A}_E-\nabla\Phi_E-\frac{\partial\mathbf{A}'_E}{\partial t}$$

At this point the potentials $\mathbf{A}_B$, $\mathbf{A}_E$, $\mathbf{A}'_B$, $\mathbf{A}'_E$, $\Phi_B$, and $\Phi_E$ are all independent and unknown. We therefore have 14 unknowns and need 14 independent linear dynamical differential equations to find a unique solution. But Maxwell's equations only contain six dynamical equations in six unknowns. By introducing more unknowns through our definition, we must also introduce more equations to ensure uniqueness. We are free to choose any new equations we want because they will have no impact on the physics as expressed in the electric and magnetic fields.

Inserting these expansions into Maxwell's equations in vacuum, we find:

$$\nabla^2 \Phi_E + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A}'_E = -\frac{\rho}{\epsilon_0}$$
$$\nabla^2 \Phi_B + \frac{\partial}{\partial t} \nabla \cdot \mathbf{A}'_B = 0$$
$$\nabla(\nabla \cdot \mathbf{A}_E) - \nabla^2 \mathbf{A}_E - \frac{\partial}{\partial t} \nabla \times \mathbf{A}'_E = -\frac{\partial}{\partial t} \nabla \times \mathbf{A}_B + \frac{\partial}{\partial t} \nabla \Phi_B + \frac{\partial^2 \mathbf{A}'_B}{\partial t^2}$$
$$\nabla(\nabla \cdot \mathbf{A}_B) - \nabla^2 \mathbf{A}_B - \frac{\partial}{\partial t} \nabla \times \mathbf{A}'_B = \mu_0 \mathbf{J} + \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \times \mathbf{A}_E - \frac{1}{c^2} \frac{\partial}{\partial t} \nabla \Phi_E - \frac{1}{c^2} \frac{\partial^2 \mathbf{A}'_E}{\partial t^2}$$

We now have eight equations and 14 unknowns. The first two equations can no longer be considered initial conditions because of the presence of the time derivative. We added the extra unknowns externally through our definition, so we are free (and required) to add any extra equations we want to in order to get a unique solution. Our choice of additional equations will have no effect on the end form of the **E** and **B** fields or on the physics, because extra unknowns are purely an artifact of the way we defined the potentials. For a unique solution, we need more initial/boundary condition equations, and 6 more dynamical equations (14 total minus the 8 already present) to add to the system of linear equations. There is no “right” set of equations to add, as they all lead to the same physical results. Note that some of the equations become trivially satisfied, so that one is tempted to discard them. But in the interest of having *n* equations in *n* unknowns, let us track all of them. We first need initial conditions specifying the divergence of the vector potentials as well as the other usual boundary conditions . One common choice (known as the Coulomb gauge) is the set of trivial conditions

$$\nabla \cdot \mathbf{A}_B = 0, \ \nabla \cdot \mathbf{A}_E = 0, \ \nabla \cdot \mathbf{A}'_B = 0, \ \nabla \cdot \mathbf{A}'_E = 0, \ \Phi_B(\text{on } S) = 0, \ \mathbf{A}'_B(t=0) = 0, \ \frac{\partial \mathbf{A}'_B}{\partial t}(t=0) = 0$$

These equations are only initial/boundary conditions. We still need to add 6 more dynamical equations for a unique solution. Because we are free to choose any equations, the trivial choices lead to the most compact final forms and are therefore the most desirable and the most traditional. The traditional choice of six additional equations to add is:

$$\mathbf{A}_E = 0$$
$$\mathbf{A}_B = \mathbf{A}'_E$$

Both of these expressions are vector expressions in three components, so they count as six equations. By adding these 6 equations (and the appropriate initial/boundary conditions) to the Maxwell equations, we therefore have 14 equations in 14 unknowns. With 14 equations and 14 unknowns and sufficient initial/boundary conditions, we therefore have a unique solution.

Note that because the additional equations and initial conditions chosen here are so trivial, Maxwell's equations in potential form seems to quickly collapse to four meaningful equations in four unknowns. For this reason, it may be tempting to claim that there is redundancy in Maxwell's equations, because we are able to go from six equations in six unknowns in the field representation to four equations in four unknowns in the potentials representation. But the truth is that we go to 14 equations in 14 unknowns in the potentials representation. With a clever choice of additional equations, most of these equations are trivial and therefore do not need to be used when solving a physics problem. But from a mathematical standpoint, all 14 equations are necessary for a unique solution in the potentials representation, and this why Maxwell's equations are not redundant.

In summary, for one particular choice of additional equations (the Coulomb gauge), Maxwell's equations in complete form in the potentials representation are:

Initial conditions:

$$\nabla\cdot\mathbf{A}_B=0\,,\ \nabla\cdot\mathbf{A}_E=0\,,\ \nabla\cdot\mathbf{A}'_B=0\,,\ \nabla\cdot\mathbf{A}'_E=0\,,\ \Phi_B(\text{on } S)=0\,,\ \mathbf{A}'_B(t=0)=0\,,\ \frac{\partial\mathbf{A}'_B}{\partial t}(t=0)=0$$

System of equations:

$$\mathbf{A}_E=0$$

$$\mathbf{A}_B=\mathbf{A}'_E$$

$$\nabla^2\Phi_E+\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}'_E=-\frac{\rho}{\epsilon_0}$$

$$\nabla^2\Phi_B+\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}'_B=0$$

$$\nabla(\nabla\cdot\mathbf{A}_E)-\nabla^2\mathbf{A}_E-\frac{\partial}{\partial t}\nabla\times\mathbf{A}'_E=-\frac{\partial}{\partial t}\nabla\times\mathbf{A}_B+\frac{\partial}{\partial t}\nabla\Phi_B+\frac{\partial^2\mathbf{A}'_B}{\partial t^2}$$

$$\nabla(\nabla\cdot\mathbf{A}_B)-\nabla^2\mathbf{A}_B-\frac{\partial}{\partial t}\nabla\times\mathbf{A}'_B=\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\times\mathbf{A}_E-\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\Phi_E-\frac{1}{c^2}\frac{\partial^2\mathbf{A}'_E}{\partial t^2}$$

*Maxwell Equations in Complete, Potentials From (Coulomb Gauge)*

Note that: $\mathbf{B}=\nabla\times\mathbf{A}_B-\nabla\Phi_B-\frac{\partial\mathbf{A}'_B}{\partial t}$ and $\mathbf{E}=\nabla\times\mathbf{A}_E-\nabla\Phi_E-\frac{\partial\mathbf{A}'_E}{\partial t}$

These equations can be inserted into each other in the usual away to arrive at mostly uncoupled equations:

Initial conditions:

$$\nabla\cdot\mathbf{A}_B=0\,,\ \nabla\cdot\mathbf{A}_E=0\,,\ \nabla\cdot\mathbf{A}'_B=0\,,\ \nabla\cdot\mathbf{A}'_E=0\,,\ \Phi_B(\text{on } S)=0\,,\ \mathbf{A}'_B(t=0)=0\,,\ \frac{\partial\mathbf{A}'_B}{\partial t}(t=0)=0$$

System of equations:

$$\mathbf{A}_E=0$$

$$\mathbf{A}_B=\mathbf{A}'_E$$

$$\Phi_B=0$$

$$\mathbf{A}'_B=0$$

$$\nabla^2\Phi_E=-\frac{\rho}{\epsilon_0}$$

$$\nabla^2\mathbf{A}_B-\frac{1}{c^2}\frac{\partial^2\mathbf{A}_B}{\partial t^2}=-\mu_0\mathbf{J}+\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\Phi_E$$

*Mostly-Uncoupled Maxwell Equations in Complete, Potentials From (Coulomb Gauge)*

Note that: $\mathbf{B}=\nabla\times\mathbf{A}_B$ and $\mathbf{E}=-\nabla\Phi_E-\frac{\partial\mathbf{A}_B}{\partial t}$

It becomes obvious in this mostly-uncoupled form, that only the last two equations are useful: the Poisson equation for the electrostatic potential and the wave equation for the magnetic vector potential. But from a mathematical standpoint, there are still 14 equations in 14 unknowns present and needed in order to have a unique solution. All 14 equations and all boundary conditions are needed to completely define the fields in terms of potentials in the most general way but still have a unique solution. If all the boundary conditions and trivial equations are ignored, Maxwell's equations in potential form in the Coulomb gauge acts like 4 equations in 4 unknowns. This is very useful to solve problems, but it does not imply that the Coulomb gauge is special or that the original Maxwell's equations contained redundancies.