

Complex functions, single and multivalued.

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1 Single-valued functions

All of the following functions can be defined via power series which converge for the entire complex plane, and hence they are called *entire* functions, being analytic everywhere:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots, \quad (1)$$

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \quad (2)$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z}), \quad (3)$$

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \quad (4)$$

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}). \quad (5)$$

Along the real axis these all reduce to the usual real series.

The following converge and are thus analytic for $|z| < 1$:

$$\begin{aligned} \log(1+z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n \\ &= z - \frac{1}{2}z^2 + \frac{1}{3}z^3 - \frac{1}{4}z^4 + \dots, \end{aligned} \quad (6)$$

and

$$\begin{aligned} (1+z)^\alpha &= \sum_{n=0}^{\infty} \binom{\alpha}{n} z^n \\ &= 1 + \alpha z + \frac{\alpha(\alpha-1)}{1 \cdot 2} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{1 \cdot 2 \cdot 3} z^3 + \dots \end{aligned} \quad (7)$$

All of these above are *single-valued*, for each z , there is one $f(z)$.

2 Multivalued functions

A *multivalued* function or *multifunction* $f(z)$ has two or more distinct values for each value of z . Important multivalued functions are

$$\sqrt{z}, \sqrt[n]{z}, \sqrt{(z-a)(z-b)}, \log z, \\ z^\alpha, \sin^{-1} z, \text{ and } \cos^{-1} z.$$

To use multivalued functions, one must pick out a branch in some region R where the functions is single-valued and continuous. This is done with *cuts* and *Riemann sheets*. Probably the simplest case is

$$w = \sqrt{z}. \tag{8}$$

If we write

$$z = |z|e^{i\theta} = re^{i\theta}, \tag{9}$$

the square root has two branches:

$$w_1 = r^{1/2}e^{i\frac{\theta}{2}}, \tag{10}$$

$$w_2 = -r^{1/2}e^{i\frac{\theta}{2}} = r^{1/2}e^{i\frac{\theta+2\pi}{2}}. \tag{11}$$

If you can stay on one branch, $w = \sqrt{z}$ is analytic except at $z = 0$.

Now think about what happens if we go around the point $z = 0$ in the complex plane. Start at some point $z = re^{i\theta}$ with the branch w_1 , and go along any closed curve that goes *once around the point* $z = 0$. Then $r \rightarrow r$, $\theta \rightarrow \theta + 2\pi$. As z moves, w_1 changes continuously, and when z returns to the original point, w_1 has changed to w_2 . If we go around again, $w_2 \rightarrow w_1$. Notice that

1. The variation of w is continuous through both branches, returning continuously to the first after two loops, and
2. the point $z = 0$ is special.

Any other path not including $z = 0$ has ordinary single-valued behavior: $r \rightarrow r$, $\theta \rightarrow \theta$, so $w_1 \rightarrow w_1$ or $w_2 \rightarrow w_2$.

The point $z = 0$ is a type of singularity called a *branch point* and is very different from a pole. The function \sqrt{z} is *unavoidably* double valued in any region that includes $z = 0$ as an interior point. If z goes on a circuit around $z = 0$, w changes to $-w$. If the branch point is inside the region there is no way to separate w_1 from w_2 . However, if we stay in a region that does not include the branch point, then as z moves around w_1 does *not* change into w_2 .

How do we avoid this? One way is simply to promise to never go all the way around the branch point. More formally this means to restrict z to a region R which excludes the branch point $z = 0$. Imagine an arbitrary region R that does not include $z = 0$ (see figure 1). Expand this region in such a way as to leave out $z = 0$, this leaves a “cut” in the plane around the branch point and extending off to infinity. This cut runs along some curve from $z = 0$ to infinity

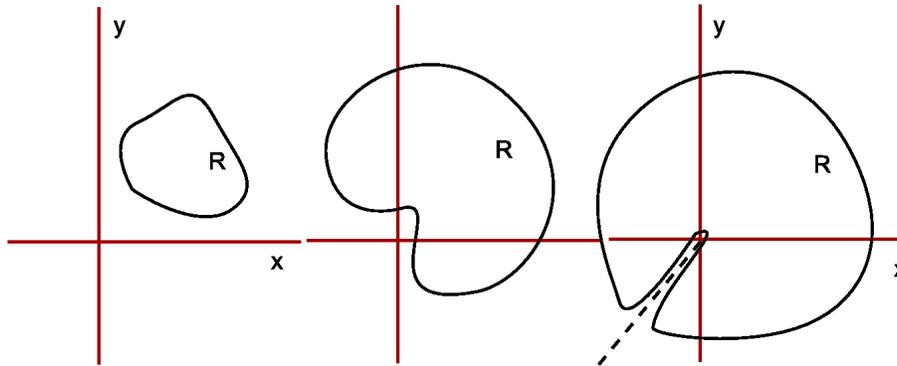


Figure 1: Enlarging the domain of a function to make a cut for a branch point at $z = 0$. The dotted line is the eventual branch cut.

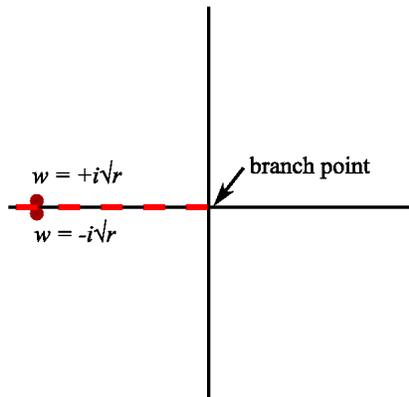


Figure 2: The standard branch cut for $w = \sqrt{z}$. The red dashed line is the cut.

in some arbitrary direction. This is a “branch cut.” The shape and direction of the curve are not fixed, they are chosen for convenience in the problem at hand.¹

Suppose we choose the negative real axis for the cut. Then we define \sqrt{z} to be

$$\sqrt{z} = r^{1/2} e^{i\theta/2}, \quad -\pi \leq \theta < \pi. \quad (12)$$

If we consider a point just above the cut on the negative real side, it has the value $r^{1/2} e^{i\pi/2} = ir^{1/2}$. On the bottom of the cut it has the value $r^{1/2} e^{-i\pi/2} = -ir^{1/2}$, so there is a discontinuity across the cut. See figure 2. So we define the function as never crossing it. If the cut is not crossed, it remains analytic.

The cut could be taken somewhere else, such as along the positive real axis,

¹There are fairly standard conventions for branch cuts in programming languages which use complex numbers, such as FORTRAN, Mathematica, Python, the C and C++ math libraries, etc.

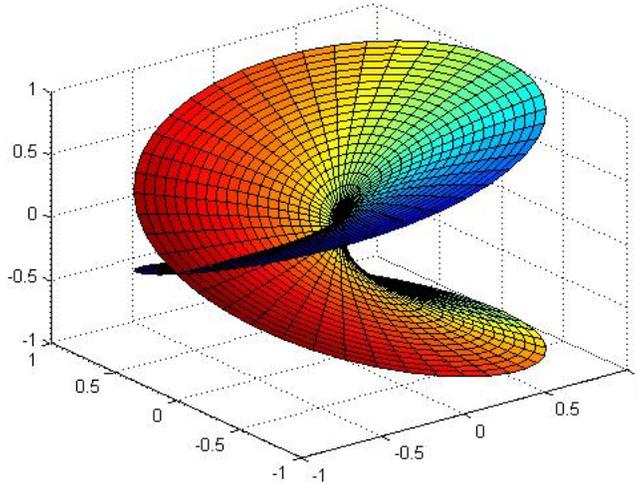


Figure 3: The Riemann sheet for \sqrt{z} . From Wikipedia.

and/or one could use the other branch as the standard. If one uses a different branch or cut, a different value will be obtained in some parts of the plane. This is often a source of trouble in translating algorithms that use complex functions from one computer language to another.

Other functions are similar but more complicated. $\sqrt[n]{z-a}$ has its branch point at $z = a$, but is otherwise identical to \sqrt{z} . $\sqrt[n]{z}$, $n = 3, 4, 5, \dots$ has a branch point at $z = 0$ and is n valued:

$$\sqrt[n]{z} = \sqrt[n]{r} e^{i \frac{\theta + 2\pi k}{n}}, \quad k = 0, 1, 2, \dots, n-1. \quad (13)$$

This can be made single-valued with a cut such as the one used with \sqrt{z} , and one of the branches:

$$\sqrt[n]{z} = \sqrt[n]{r} e^{i \frac{\theta}{n}}, \quad -\pi \leq \theta < \pi. \quad (14)$$

3 Stitching the plane back together: Riemann sheets

Riemann devised a way to keep both branches of \sqrt{z} while maintaining a continuous single-valued function. Imagine separating the z plane into two sheets connected along the cut. The independent variable z has the same value at two points, one on the “top” sheet, and one on the “bottom.” On the top sheet \sqrt{z} has the values associated with one branch, on the bottom sheet, it has the values associated with the other branch. See figure 3 If the cut is along the

negative real axis, then as θ goes through π , z goes down through the cut on the bottom sheet, then back up as it goes through 3π . In a complete traversal of both sheets around the branch point θ increases by 4π , $\theta/2$ by 2π , and \sqrt{z} comes back to its starting value. The cut is arbitrary, and you can disregard the intersection of the sheets along the cut. The branch point gives the surface this topology.

If a function has more values, it has more sheets: $\sqrt[n]{z}$ has n sheets connected at the cut. If the function has two branch points like $\sqrt{(z-a)(z-b)}$, a common choice is to take the cut along the line between $z = a$ and $z = b$, and gives two sheets connected along this cut. With multiple branch points, Riemann sheets or surfaces end up with all sorts of strange and wonderful topological properties.

4 An infinitely multivalued function: the logarithm

The function $\log z$ is the inverse of the exponential. So if

$$z = e^w, \tag{15}$$

then

$$w = \log z. \tag{16}$$

By considering the power series for e^w

$$\frac{dz}{dw} = e^w = z \tag{17}$$

so

$$\frac{dw}{dz} = \frac{d}{dz} \log z = \frac{1}{z}. \tag{18}$$

Since it has a derivative, the logarithm is analytic (again in one branch, or on a Riemann sheet).

The logarithm has an infinite number of branches. We can write from $z = e^w$

$$z = re^{i\theta} = re^{i\theta + i2\pi n} = e^{\ln r + i\theta + i2\pi n} \tag{19}$$

where $n = 0, \pm 1, \pm 2, \dots$, any integer. So the logarithm is

$$\log z = \ln r + i\theta + i2\pi n. \tag{20}$$

Every time z moves in a closed curve around the branch point $z = 0$, θ increases by 2π and we go from one branch to another, and this can go on forever. θ goes through multiples of 2π , and the infinite set of values of the logarithm differ by $2\pi ni$. The branch point $z = 0$ is of infinite order for $\log z$.

We can get a single valued function by picking a cut, such as the negative real axis:

$$\log z = \ln r + i\theta, \quad -\pi \leq \theta < \pi. \tag{21}$$

At two points z_1, z_2 on either side of the cut

$$\log z_1 = \ln r - i\pi, \quad (22)$$

$$\log z_2 = \ln r + i\pi. \quad (23)$$

The discontinuity across the cut is $2\pi i$.

To make this continuous via Riemann surface, it has an infinite number of sheets attached at $z = 0$, spiraling like a helix or screw, if you visualize it in a third dimension.

Similarly, z^α has an infinite order branch point if α is an irrational number. If $\alpha = p/q$ is a rational number (p, q integers) then it has q branches or sheets. The formula

$$z^\alpha = e^{\alpha(\ln r + i\theta)} \quad (24)$$

helps to show this.

5 References

This version is based on section 10.4 of Wyld, *Mathematical Methods for Physics*.

Other good things on the Web are the Wikipedia articles of “Complex plane” and “Riemann surface” as of 2007-03-27.